New Topics in Information and Control Systems involving CRT

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Outline

- We briefly introduce two new research topics
    1. Coding: Color multiset coding
    2. Control : Attention control
- Both topics have connections with CRT:

Recall the Chinese Remainder Theorem (CRT)

- **Theorem**: Let $n_1, \ldots, n_k$ be pairwise coprime positive numbers and $a_1, \ldots, a_k$ any integers. Then
$$x \equiv a_1 \mod n_1$$
$$\vdots$$
$$x \equiv a_k \mod n_k$$
has a solution x uniquely determined modulo $N = n_1, \ldots, n_k$. In fact,
- $x = \sum_{i=1}^{k} a_i m_i y_i$ where $m_i = N/n_i$, $y_i = m_i^{-1} \mod n_i$
- CRT defines a 1-1 onto mapping, $\theta : \mathbb{Z}_N \to \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$

Depicting the two representations

- For example, $\mathbb{Z}_{pq}$ and $\mathbb{Z}_p \times \mathbb{Z}_q$
- If $p = 3, q = 4$

$$p = 3 \quad \begin{bmatrix} 0 & 9 & 6 & 3 \\ 4 & 1 & 10 & 7 \\ 8 & 5 & 2 & 11 \end{bmatrix}$$

$$q = 4$$

- Another way to represent the correspondence

$$\begin{matrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{matrix}$$

Can we generalize to non-coprime cases?

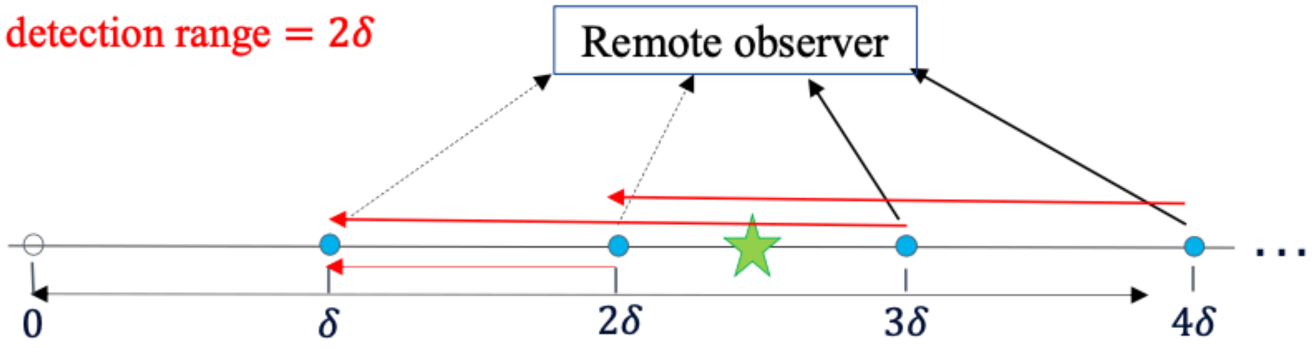
- Yes, but then θ is neither 1-to-1 nor onto. E.g. for 10 and 12.
- Can make θ 1-to-1 by restricting to $\mathbb{Z}_{60}$ but cannot be onto $\mathbb{Z}_{10} \times \mathbb{Z}_{12}$
- Explains why there is only have 60 years in one 甲子.

甲	1		1		1			1			1	
乙		1		1		1			1			1
丙	1		1		1		1			1		
丁		1		1		1		1			1	
戊					1		1		1			1
己	1		1			1		1		1		
庚		1		1			1		1		1	
辛			1		1			1		1		1
壬	1					1			1		1	
癸		1		1			1			1		1
	子	丑	寅	卯	辰	巳	午	未	申	酉	戌	亥

First Problem:
Color Multiset Coding

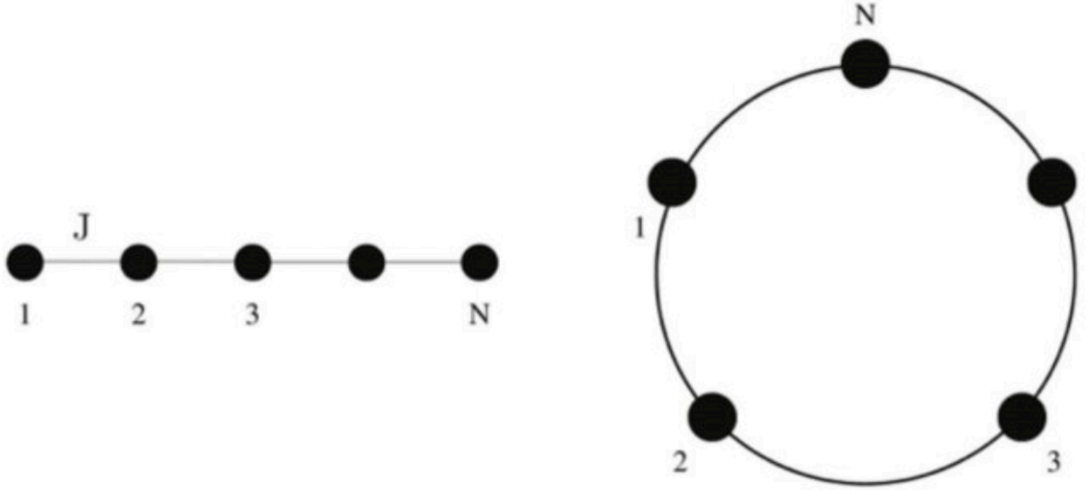
A motivating problem: Object tracking

- Consider a sensor grid with sensor detection range $m/2$, so that exactly m sensors at triggered at one time.

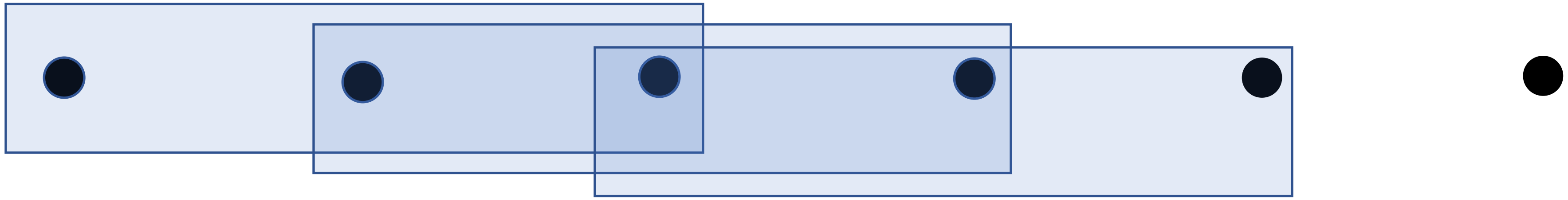


- Each sensor is equipped with an LED light, which is turned on when the object is detected.
- We want to design a color sequence so that a remote observer to track the position of an object moving over the grid

Mathematic setting



- Let G be a cyclic 1D grid of size N: $N+1$ is identified with 1, and so on.
- (A flat grid can be solved similarly.)
- A block of size m is a subset of m consecutive grid points: $(i, i+1, \ldots, i+m-1)$ (The arithmetic is modulo N)
- The block starting at i can be identified with the point i and vice versa



$3 - blocks$

Object tracking via Color Coding

- Let C be a set of k colors.
- Question: Can we color the grid points of G to *uniquely* identify all the blocks by the set of block colors?
- First version:
- The traditional formulation: the ordering of the colors is known.

- The 2-blocks defined by this color sequence are:

$$(B,B),(B,R),(R,B),(B,Y),(Y,Y),(Y,B)$$

Multiset formulation

- What if the observer is too far away that the spatial resolution is unable to convey the ordering or say wireless frequencies are used?
- Astronomy tells us there is much information in the frequency spectrum and intensity
- What if we can observe the colors and their numbers?
- We can represent these as *multisets*： An unordered collection of elements where an element can appear multiple times
- E.g. $\{R, R\}, \{R, B\}$ Note $\{R, R\} \neq \{R\}$
- There is data compression: $(R, B), (B, R)$ yield the same multiset $\{R, B\}$

Color multiset coding

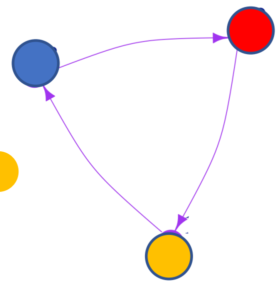
- Let G be a cyclic 1D grid of size M with block size m
- Let C be a set of k colors.
- A color sequence is a mapping from G to C.
- Example: $M = 6, k = 3$

- The 2-block color sets are : $\{B, B\}, \{B, R\}, \{R, R\}, \{R, Y\}, \{Y, Y\}, \{Y, B\}$
- Note these are multisets
- And they are *distinct multisets!*
- Call this a m *−distinguishable color multiset code*

Some simple m −distinguishable solutions

- $m = 1, M = 6$ clearly we need 6 colors

- For $m = 2, M = 6$, $bbggrr$ we only need 3 colors
- In fact, for k colors draw a complete k vertices graph (with self looping edges) each edge represents an order 2 color multiset. The problem can be solved by finding an Eulerian circuit.



Self loops not shown

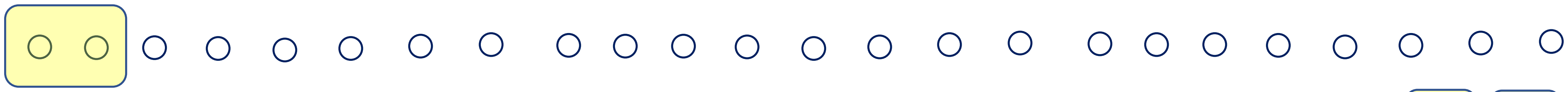
- How about larger m values?

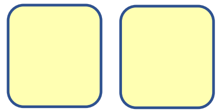
How to construct distinguishable color multiset codes in general?

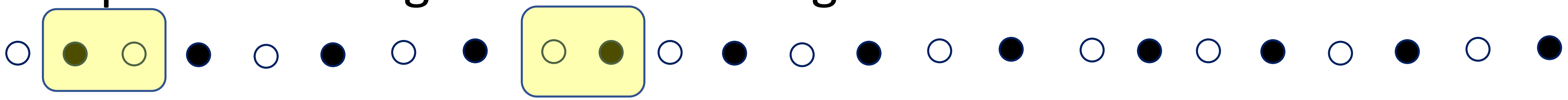
- Difficulty of finding a solution increases with m and M. Finding optimal solutions are even harder.
- For a given grid size M, the minimal number of colors is used. Represent this number by $K(M)$
- Optimal solution is known only for some **small values of $\boldsymbol{m}$**
- But asymptotic order of growth $O(K(M))$ of the optimal solution is known: $O(M^{1/m})$
- Although optimal solutions for general m are still unknown, we can construct solutions whose asymptotic order $O(K(M))$ is the same as the optimal solutions $O(M^{1/m})$

Construction for general block size via sunmao construction

- Consider the case $m = 2$ with $M = 24$



- We can view the block as consisting of two size 1 sub-blocks 
- If we partition the grid into two sub-grids:



then each sub-block consists of 1 element from each sub-grids

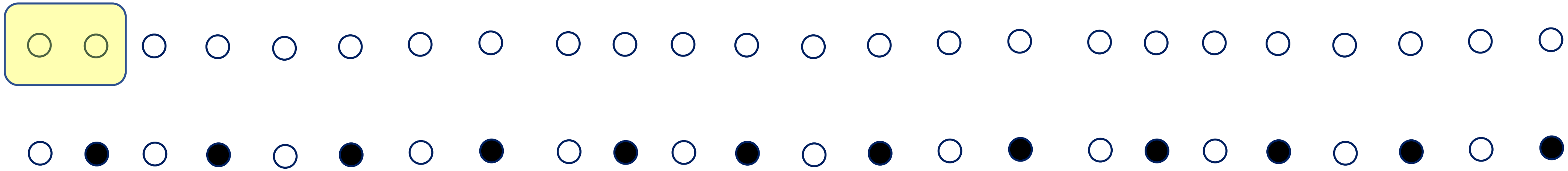
- We can construct solutions for each sub-grid with the corresponding sub-block size and piece the solutions together for the original grid.

Braid code

- *Braid code* is a class of codes based on the sunmao methodology, in which an $m-$ block is decomposed into I sub-blocks, $m = m_1 + \cdots + m_I$
- **Theorem**: Braid codes are m $-$distinguishable.
- The special case where all the decomposed sub-blocks are singletons are referred to as *unitary braid codes.*

An example of a unitary braid code

- Consider the $m = 2$ case with $M = 24$



0 1 2 3 0 1 2 3 0 1 2 3

0 1 2 3 4 5 0 1 2 3 4 5

- Note we use $p = 4 = 2 * 2$ and $q = 2 * 3$ colors, $M = 2 * 2 * \mathrm{lcm}(2,3)$
- Resultant color mapping:

0 0 1 1 2 2 3 3 0 4 1 5 2 0 3 1 0 2 1 3 2 4 3 5

- It can be proven that the derived color multiset code is 2-distinguishable.

Another braid code example

- Decompose $m = 3$ into $m_1 = 1, m_2 = 2, M = 36$
- Sub-grid 1 has size 12 and a color mapping obtained by repeating $a_1 a_2 a_3 a_4 a_5 a_6$ 2 times: $a_1 a_2 a_3 a_4 a_5 a_6$ $a_1 a_2 a_3 a_4 a_5 a_6$
- Sub-grid 2 has size 24 and a color mapping obtained by repeating

$b_1 b_1 b_2 b_2 b_3 b_3 b_4 b_4$ 3 times: $b_1 b_1 b_2 b_2 b_3 b_3 b_4 b_4$ $b_1 b_1 b_2 b_2 b_3 b_3 b_4 b_4$ $b_1 b_1 b_2 b_2 b_3 b_3 b_4 b_4$

- Combining via sunmao construction results in a 3-distinguishable mapping using 10 colors:

$$a_1 b_1 b_1 a_2 b_2 b_2 a_3 b_3 b_3 a_4 b_4 b_4 a_5 b_1 b_1 a_6 b_2 b_2$$
$$a_1 b_3 b_3 a_2 b_4 b_4 a_3 b_1 b_1 a_4 b_2 b_2 a_5 b_3 b_3 a_6 b_4 b_4$$

Braid code efficiency

- In terms of color efficiency the unitary braid code has the same asymptotic order of growth $O(K(M))$ as the optimal code.
- E.g. for $m = 2$, the minimal colors required to distinguishably code a 1D grid of size M is approximately $(2M)^{1/2}$
- Using unitary braid code, the number of colors is approximately $(4M)^{1/2}$.

- Braid code and CRT

Sub-grid 1	0		1		2		3		0		1		2		3		0		1		2		3	
Sub-grid 2		0		1		2		3		4		5		0		1		2		3		4		5
Grid formed by sunmao	**0**	**0**	**1**	**1**	**2**	**2**	**3**	**3**	**0**	**4**	**1**	**5**	**2**	**0**	**3**	**1**	**0**	**2**	**1**	**3**	**2**	**4**	**3**	**5**

$\theta_1(i)$	0	1	1	2	2	3	3	0	0	1	1	2	2	3	3	0	0	1	1	2	2	3	3	0
$\theta_2(i)$	0	0	1	1	2	2	3	3	4	4	5	5	0	0	1	1	2	2	3	3	4	4	5	5
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

- This defines a bijection mapping θ from $\mathbb{Z}_{24}$ onto $\mathbb{Z}_4 \times \mathbb{Z}_6$!

$$\theta(i) = \left(\left\lfloor \frac{i+1}{2} \right\rfloor \bmod 4, \left\lfloor \frac{i}{2} \right\rfloor \bmod 6\right)$$

- We can also use CRT to obtain a fast decoding algorithm

Color multiset code for n-dimensional grids

- It is possible to define braid code for n-dimensional grids by using *product code*
- The number of colors required for these braid codes also have the same asymptotic order as the optimal codes!
- Some references:
- C. S. Chen, Y.-H. Lo, W. S. Wong, and Y. Zhang, "Object tracking using multiset color coding," to appear in IEEE International Symposium on Information Theory and Its Applications, 2024.
- C. S. Chen, W. S. Wong, Y.-H. Lo, and T.-L. Wong, "Multiset combinatorial gray codes with application to proximity sensor networks," 2024. [Online]. Available: https://arxiv.org/abs/2410.15428
- W. S. Wong, C. S. Chen and Y.-H. Lo, "Color Multiset Codes based on Sunmao Construction" Available: http://arxiv.org/abs/2511.09070