Object Tracking Using Multiset Color Coding

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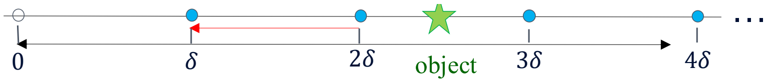
Outline

Object Tracking Problem

- An object randomly appears on a line of length L.
- Time is divided into descrete time slots of duration h.
- At the beginning of each time slot, we want to determine the position of the object, with upside precision δ.
  - the system says it is located at 2δ, it means: in $[2\delta, 3\delta)$
  - δ: tracking accuracy



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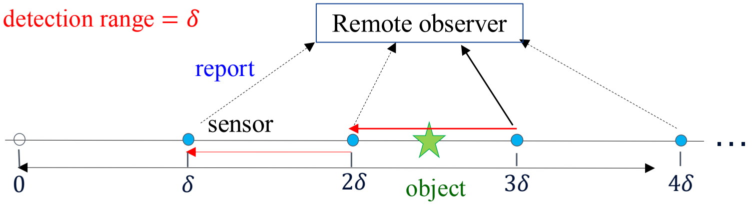

length L with $3\delta < L \leq 4\delta$ (e.g., $N = 4$)

A straightforward solution:

- put one sensor at each position located $\delta, 2\delta, \ldots, \lceil L/\delta \rceil \delta$
- each sensor detects objects to its left with detection range δ
- each sensor reports the location of the detected object to a remote observer

$N \triangleq \lceil L/\delta \rceil$ unique IDs are necessary for the N sensors, each of which requires $\log_2 N$ bits to represent.

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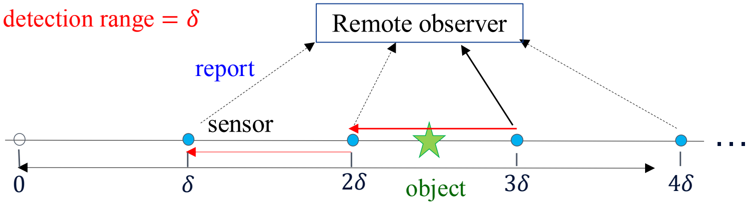

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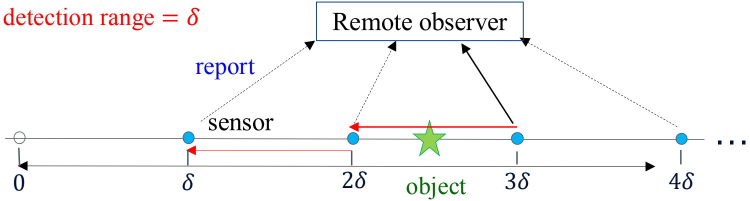

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Straightforward Protocol

$N \triangleq \lceil L/\delta \rceil$ unique IDs are necessary for the N sensors, each of which requires $\log_2 N$ bits to represent.

- Time is divided into descrete time slots of duration h.
- Assume the communication channel is with date rate R (bits/time-unit).
- The straightforward protocol works whenever

$$\log_2 N = \log_2 \lceil L/\delta \rceil \leq Rh. \qquad (*)$$

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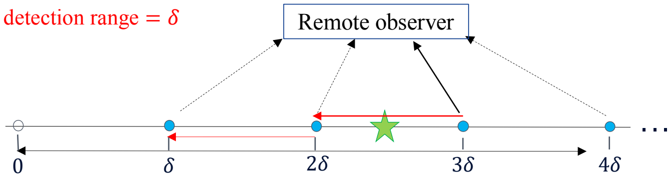
- $(*)$ does not hold if $L > \delta 2^{Rh}$
- If the detection range $> \delta$, the straightforward protocol seems a bit lavish.

Object Tracking Problem



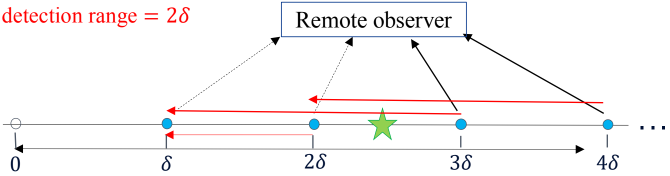
- Some IDs may be reused to reduce the number of IDs.

Multiset Color Coding Problem

Assume the detection range is $m\delta$, for $m \in \mathbb{N}$, where δ is the tracking accuracy. Find a set of $k \leq N$ IDs and select one ID for each sensor so that, the combinations of m consecutive IDs for all intervals $[(j-1)\delta, j\delta]$ are all distinct (distinguishable).

Objective: For given N and m, minimize the value k.

Object Tracking Problem



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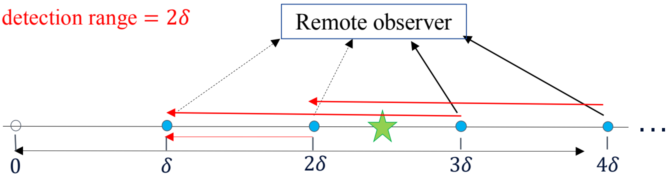

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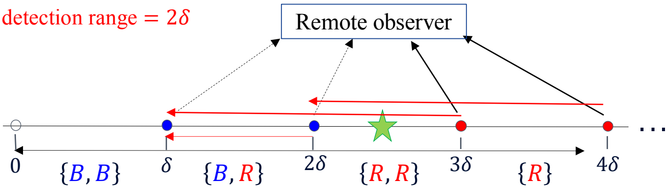

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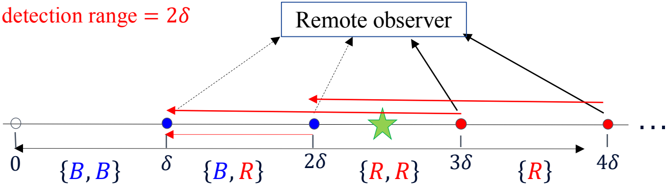

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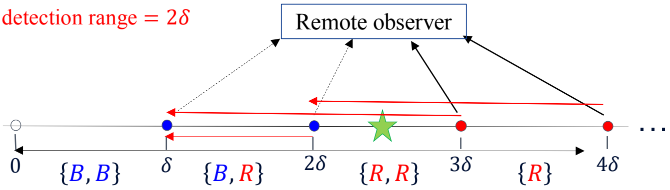

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Mathematical Modeling

Notation and Definitions

- Let $[k] \triangleq \{1, 2, \ldots, k\}$, a set of k colors.
- Consider a sequence $S = s_0 s_1 \cdots s_{N-1}$ in which $s_i \in [k]$.
- Let $S_t(m)$ denote the **multiset** $\{s_t, s_{t+1}, \ldots, s_{t+m-1}\}$.

Definition

- m-distinguishable: multisets $S_t(m)$, $t \in \{0, 1, \ldots, N-m\}$, are all distinct
- cyclic m-distinguishable: multisets $S_t(m)$ are all distinct for $t \in \mathbb{Z}_N$

Example. $S = 2113\ 3212$

- S is 3-distinguishable, but not cyclic 3-distinguishable

An equivalent problem

For given m and sequence length N, minimize the number of colors k.

$$\Updownarrow$$

For given m and color size k, maximize the sequence length N $\longleftarrow N_m(k)$

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Lower bound on $N_m(k)$

Proposition

For given positive integers m and k, one has

$$N_m(k) \leq \binom{k+m-1}{m} + m - 1.$$

Proof.

- $S = s_0 s_1 \cdots s_{N-1}$
- $N - m + 1$ multisets $S_t(m)$, $0 \leq t \leq N - m$, are all distinct
- $S_t(m) = \{1^{e_1}, \ldots, k^{e_k}\}$, where e_s is the multiplicity of the element s
- $e_1 + e_2 + \cdots + e_k = m$ with each $e_s \geq 0$
- The number of non-negative integral solutions for above is $\binom{k+m-1}{m}$

$$\Rightarrow N - m + 1 \leq \binom{k+m-1}{m}.$$

Main Results

Universal Cycles

Definition (Universal cycles, Ucycles)

A (k,m)-Ucycle is a cyclic m-distinguishable sequence S on $[k]$ in which

1. there is no repeated elements in any $S_t(m)$, and
2. every m-subset of $[k]$ appears exactly once as $S_t(m)$ for some t.

Some facts:

- A (k,m)-Ucycle is of length $\binom{k}{m}$.
- A (k,m)-Ucycle exists, then $m \mid \binom{k-1}{m-1}$

Example. $S = 12345\ 13524$ is a $(5,2)$-Ucycle

$\Rightarrow 12345\ 13524$ is a cyclic 2-distinguishable sequence

$\Rightarrow 12345\ 13524\ \mathbf{1}$ is a 2-distinguishable sequence

F. Chung, P. Diaconis, and R. Graham, "Universal cycles for combinatorial structures," *Discrete Math.*, vol. 110, pp. 43–59, 1992.

Universal Cycles

Theorem (Glock–Joos–Kühn–Osthus, 2020)

For every $m \in \mathbb{N}$, there exists $n_o \in \mathbb{N}$ such that for all $k \geq n_o$, there exists a (k, m)-Ucycle whenever m divides $\binom{k-1}{m-1}$.

Corollary (–, 2024)

Let $m \in \mathbb{N}$ and k be a sufficiently large integer. Then, if m divides $\binom{k-1}{m-1}$, then

$$\binom{k}{m} + m - 1 \leq N_m(k) \leq \binom{k+m-1}{m} + m - 1.$$

Therefore, we have: $N_m(k) \approx \dfrac{k^m}{m!}$.

S. Glock, F. Joos, D. Kühn, and D. Osthus, "Euler tours in hypergraphs," *Combinatorica*, vol. 40, pp. 679–690, 2020.

Main Result I: color-coding gain

The number of bits to represent an ID :

- in "straightforward protocol", it is $\log_2 N$
- in "multiset color coding protocol", it is $\log_2 k$

Fix m. For large enough k, we have $N \approx \frac{k^m}{m!}$.

$$\begin{aligned}\log_2 N &\approx m\log_2 k - \log_2 m! \\ &= m\log_2 k - (m\log_2 m - m\log_2 e + O(\log_2 m))\end{aligned}$$

$$\Rightarrow \log_2 k \approx \frac{\log_2 N}{m} + \log_2 m - \log_2 e + \frac{c\log_2 m}{m}$$

As $N \to \infty$, we get

$$\frac{\log_2 k}{\log_2 N} = \frac{1}{m}.$$

Mcycles

Definition (Universal cycles for multisets, Mcycles)

A (k,m)-Mcycle is a cyclic m-distinguishable sequence S in which

- every m-multiset of $[k]$ appears exactly once as $S_t(m)$ for some t.

Some facts:

- A (k,m)-Mcycle is of length $\binom{k+m-1}{m}$.
- A (k,m)-Mcycle exists, then $m \,\Big|\, \binom{k+m-1}{m}$
- A $(k,\mathbf{2})$-Mcycle is an Eulerian circuit of K_k^{ℓ} (with loops)

Theorem (Hurlbert–Johnson–Zahl, 2009)

For $k \geq 4$ with $k \equiv 1, 2 \pmod 3$, a $(k,3)$-Mcycle exists.

G. Hurlbert, T. Johnson, and J. Zahl, "On universal cycles for multisets," *Discrete Math.*, vol. 309, pp. 5321–5327, 2009.

Main Results II: constructive methods

Corollary

One has

$$N_m(k) = \begin{cases} \binom{k+1}{2} + 1 & \text{if } m = 2 \text{ and } k \equiv 1 \pmod{2}, \\ \binom{k+2}{3} + 2 & \text{if } m = 3 \text{ and } k \equiv 1, 2 \pmod{3}. \end{cases}$$

Theorem (–, 2024)

For the missing cases, we have

$$N_m(k) \geq \begin{cases} \binom{k+1}{2} - \frac{k}{2} + 1 & \textit{if } m = 2 \textit{ and } k \equiv 0 \pmod{2}, \\ \binom{k+2}{3} - \frac{k}{3} + 2 & \textit{if } m = 3 \textit{ and } k \equiv 0 \pmod{3}. \end{cases}$$

Remark. The above result is optimal in our subsequent paper.

C. S. Chen, W. S. Wong, Y.-H. Lo, and T.-L. Wong, "Multiset combinatorial Gray codes with application to proximity sensor networks," *arXiv:2410.15428*

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Main Results II: constructive methods

Theorem ($m = 3$)

For k a multiple of 3, one has

$$N_3(k) \geq \binom{k+2}{3} - \frac{k}{3} + 2.$$

<u>*Construction.*</u>

- By induction on k.
  - $k = 3$: 111222333
  - $k = 6$: 11122 23331 16631 55224 53532 44336
    21414 62625 14365 55444 6665
- Assume on $[k-3]$, it is of the form $S'T'$
- For $[k]$, the obtained sequence is of the form $WXYZ$, where
  - $W = S'Y'$
  - X is obtained from T' by $k-5 \mapsto k-2$, $k-4 \mapsto k-1$, and $k-3 \mapsto k$.
  - Y and Z are two special patterns, according to the parity of k.

Concluding Remarks

Conclusion

- Consider the tracking problem of an object that randomly appears on a line of a fixed length.
- Propose a newly defined multiset color coding protocol with nice design which employs the minimal number of bits for labeling each sensor (i.e., representing their IDs).
- The number of bits needed can be reduced by a factor $1/m$ by the proposed scheme.
- Some constructive methods are given.

Future Works

1. More (optimal) constructions for m-distinguishable sequences.
2. Any other constructive methods for universal cycles.
3. Extension to 2D grids or higher dimensional cases.

Definition (–, 2025+)

Consider a 2D grid of size $M \times N$ and a labeling Φ of grid points with elements in $[k]$. The labeling Φ is called (m, n)-distinguishable if multisets $S_{m,n}(x_0, y_0)$ are all distinct, where

$$S_{m,n}(x_0, y_0) \triangleq \{\Phi(x_0 + i, y_0 + j) : 0 \leq i < m, 0 \leq j < n\}.$$

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Thank you for your listening