Robot positioning using torus packing for multisets

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Robot on a line

de Bruijn sequence size $n = 27$
$k = 3$ colors
window size $m = 3$
all windows contain distinct color patterns
wraps around.

0 1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0

0 1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0
1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0 0
1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0 0 1

Reaches the upper bound $n \leq k^m$.

Robot on a line

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0	1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0

0	1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0
1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0	0
1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0	0	1

Reaches the upper bound $n \leq k^m$.

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0 1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0

0 1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0
1 1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0 0
1 1 2 1 0 1 2 0 1 0 2 1 1 0 0 2 2 1 2 2 2 0 2 0 0 0 1

Reaches the upper bound $n \leq k^m$.

Robot on a line

de Bruijn sequence size $n = 27$
$k = 3$ colors
window size $m = 3$
all windows contain distinct color patterns
wraps around.

0	1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0

0	1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0
1	1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0	0
1	1	2	1	0	1	2	0	1	0	2	1	1	0	0	2	2	1	2	2	2	0	2	0	0	0	1

Reaches the upper bound $n \leq k^m$.

Robot on a torus

de Bruijn torus size $n = 4$
$k = 2$ colors
window size $m = 2$
all windows contain distinct color patterns
wraps around.

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

Reaches the upper bound $n^2 \leq k^{m^2}$.

Robot on a torus

de Bruijn torus size $n = 4$
$k = 2$ colors
window size $m = 2$
all windows contain distinct color patterns
wraps around.

$$\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}$$

Reaches the upper bound $n^2 \leq k^{m^2}$.

Multiset vs sequence

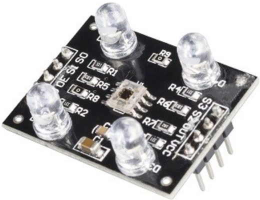
Measuring an ordered pattern requires sophisticated hardware

- camera
- directional antenna

Measuring a multiset is cheaper

- color sensor
- RFID tag, used in retail for inventory management, self checkout and theft prevention

Cost increases with the number of colors / tag identifiers.



Our problem

Color a d dimensional torus of size n with k colors such that no two windows of size m contain the same multiset of colors.

0	2	1	1	3
3	0	3	2	3
1	3	2	2	1
0	0	2	2	1
0	0	2	1	0

dimension $d = 2$, size $n = 5$, number of colors $k = 4$, window size $m = 2$.

$\{0, 0, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{1, 1, 2, 3\}$	$\{1, 2, 3, 3\}$	$\{0, 3, 3, 3\}$
$\{0, 1, 3, 3\}$	$\{0, 2, 3, 3\}$	$\{2, 2, 2, 3\}$	$\{1, 2, 2, 3\}$	$\{1, 1, 3, 3\}$
$\{0, 0, 2, 3\}$	$\{0, 2, 2, 3\}$	$\{2, 2, 2, 2\}$	$\{1, 1, 2, 2\}$	$\{0, 1, 1, 1\}$
$\{0, 0, 0, 0\}$	$\{0, 0, 2, 2\}$	$\{1, 2, 2, 2\}$	$\{0, 1, 1, 2\}$	$\{0, 0, 0, 1\}$
$\{0, 0, 0, 2\}$	$\{0, 1, 2, 2\}$	$\{1, 1, 1, 2\}$	$\{0, 1, 1, 3\}$	$\{0, 0, 0, 3\}$

Related works

Universal cycles / torii [Chung Graham Diaconis 1992]
Each window outputs a function of its content.

- Classic. Each possible output appears exactly once.
- Packing. No output repetition (maximize the size).
- Covering. Each output appears at least once (minimize the size).

Universal cycles for multisets. Open problem [Knuth 2011] with interesting progress [Hurlbert Johnson Zahl 2009, Blanca Godbole 2011].

Universal cycles for set. Conjecture from [Chung Diaconis Graham 1992] solved by [Glock Joos Kühn 2020].

de Bruijn torii
[MacWilliams Sloane 1976, Hurlbert Isaak 1993, 1995, Hurlbert Mitchell Paterson 1996]

Our result: torus packing for multisets

Fix the dimension $d \geq 2$ and the number of colors k.

Theorem Choose a number of colors $k = 1 \mod d$ and a window size m multiple of $k-1$. We construct a torus packing for multisets of size

$$n \sim C_k^{1/d} m^{k-1} \qquad\qquad C_k = \left(\frac{2}{k-1}\right)^{k-1}$$

Asymptotical optimality

number of windows $\leq$ number of color multisets fitting a window

$$n \leq \binom{m^d + k - 1}{k-1}^{1/d} \sim {C'_k}^{1/d} m^{k-1} \qquad\qquad C'_k = \frac{1}{(k-1)!}$$

Decoding. Constant number of arithmetic operations.

Construction for $k = 3$ colors

One special color 0.
The other colors are partionned and dedicated to a dimension.
Periodic coloring, then erase colors using the special one 0.

window size $m = 4$, x-axis uses color 1, y-axis uses color 2

1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1

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window size $m = 4$, x-axis uses color 1, y-axis uses color 2

$$\begin{array}{cccccccc} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \end{array}$$

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window size $m = 4$, x-axis uses color 1, y-axis uses color 2

1		1		1		1	
	1		1		1		1
1		1		1		1	
	1		1		1		1

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1		1		1		1	
	1		1		1		1
1		1		1		1	
	1		1		1		1

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	1		1		1		1	
		1		1		1		1
	1		1		1		1	
		1		1		1		1
dual	0	1	2	3	4	5	6	7

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	0		0		1		1	
		0		0		1		1
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0				
dual	0	1	2	3	4	5	6	7

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window size $m = 4$, x-axis uses color 1, y-axis uses color 2

	0		0		0		1	
		0		0		1		1
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	1			
dual	0	1	2	3	4	5	6	7

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	0		0		0		1	
		0		0		0		1
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	1	1		
dual	0	1	2	3	4	5	6	7

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	0		0		0		0	
		0		0		0		1
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	1	1	1	
dual	0	1	2	3	4	5	6	7

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	0		0		0		0	
		0		0		0		0
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	1	1	1	1
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	0		0		0		0	
		0		0		0		0
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	1	1	1	1
dual	0	1	2	3	4	5	6	7

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	0		0		1		1	
		0		0		1		0
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	2	2	2	1
dual	0	2	4	6	7	5	3	1

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		0		0		1		1
	0		0		1		1	
		0		0		1		0
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	2	2	2	1
dual	0	2	4	6	7	5	3	1

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window size $m = 4$, x-axis uses color 1, y-axis uses color 2

	0		0		1		1	
		0		0		1		0
	0		0		1		1	
		0		0		1		1
	0		0		1		1	
		0		0		1		0
	0		0		1		1	
		0		0		1		1
profile	0	0	0	0	2	2	2	1
dual	0	2	4	6	7	5	3	1

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One special color 0.
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Periodic coloring, then erase colors using the special one 0.

window size $m = 4$, x-axis uses color 1, y-axis uses color 2

	0	2	0	2	1	2	1	2
	2	0	2	0	2	1	2	0
	0	2	0	2	1	2	1	2
	2	0	2	0	2	1	2	1
	0	2	0	2	1	2	1	2
	2	0	2	0	2	1	2	0
	0	2	0	2	1	2	1	2
	2	0	2	0	2	1	2	1
profile	0	0	0	0	2	2	2	1
dual	0	2	4	6	7	5	3	1

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One special color 0.
The other colors are partionned and dedicated to a dimension.
Periodic coloring, then erase colors using the special one 0.

window size $m = 4$, x-axis uses color 1, y-axis uses color 2

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 2 & 0 & 2 & 1 & 2 & 1 & 2 \\
2 & 0 & 2 & 0 & 2 & 1 & 2 & 0 \\
0 & 2 & 0 & 2 & 1 & 2 & 1 & 2 \\
2 & 0 & 0 & 0 & 2 & 1 & 0 & 1
\end{array}$$

profile	0	0	0	0	2	2	2	1
dual	0	2	4	6	7	5	3	1

Construction for more colors

Example window of size 8

x-axis position coded using 1 and 2.

4	1	2	3	4	1	2	3	4	1	2	3	$\cdots$
3	4	1	2	3	4	1	2	3	4	1	2	$\cdots$
2	3	4	1	2	3	4	1	2	3	4	1	$\cdots$
1	2	3	4	1	2	3	4	1	2	3	4	$\cdots$
4	1	2	3	4	1	2	3	4	1	2	3	$\cdots$
3	4	1	2	3	4	1	2	3	4	1	2	$\cdots$
2	3	4	1	2	3	4	1	2	3	4	1	$\cdots$
1	2	3	4	1	2	3	4	1	2	3	4	$\cdots$

Construction for more colors

Example window of size 8

x-axis position coded using 1 and 2.

	1	2			1	2			1	2		$\cdots$
		1	2			1	2			1	2	$\cdots$
2			1	2			1	2			1	$\cdots$
1	2			1	2			1	2			$\cdots$
	1	2			1	2			1	2		$\cdots$
		1	2			1	2			1	2	$\cdots$
2			1	2			1	2			1	$\cdots$
1	2			1	2			1	2			$\cdots$

Construction for more colors

Example window of size 8
x-axis position coded using 1 and 2.

		0	0			0	0			0	0		$\cdots$
			1	0			0	0			0	0	$\cdots$
	0			0	2			1	2			0	$\cdots$
	0	0			0	0			0	0			$\cdots$
		1	2			0	0			0	0		$\cdots$
			1	0			1	0			0	2	$\cdots$
	0			1	2			1	2			0	$\cdots$
	0	0			0	0			0	2			$\cdots$
profile	0	0	1	0	2	0	0	0	2	1	0	1	(2)
	0	1	2	1	0	0	1	2	0	0	0	0	(1)
dual	3	5	6	5	6	.	.	.	.	.	.	.	
	7	7	6	4	3	.	.	.	.	.	.	.	

Vector sum packing

Sequence of vectors from $\{0, 1, \ldots, s\}^\ell$
window of size m
No two windows have the same vector sum

Reduction. Torus packing for multisets $\mapsto$ Cycle packing for vector sums

Example. vector size $\ell = 2$, window size $m = 2$, maximum value $s = 2$

Profile

Ideal dual	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1

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Sequence of vectors from $\{0, 1, \ldots, s\}^\ell$
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Example. vector size $\ell = 2$, window size $m = 2$, maximum value $s = 2$

Profile	0	0	2	1	0	0	2	1	0	0	2	1	0	0	2	1
Ideal dual	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1

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Example. vector size $\ell = 2$, window size $m = 2$, maximum value $s = 2$

Profile	0	0	2	1	0	0	2	1	0	0	2	1	0	0	2	1
Ideal dual	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1
	0	0	0	0	2	2	2	2	3	3	3	3	1	1	1	1

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Profile	0	0	2	1	0	0	2	1	0	0	2	1	0	0	2	1	
	0	0	0	0	0	2	0	2	0	?	?	?	?	?	?	?	Oh no!
Ideal dual	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1	
	0	0	0	0	2	2	2	2	3	3	3	3	1	1	1	1	

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Profile													
	0	0	0	0	0	2	0	2	1	2	1	0	1
dual													
	0	0	0	0	2	2	2	3	3	3	1	1	1

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Sequence of vectors from $\{0, 1, \ldots, s\}^{\ell}$
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Example. vector size $\ell = 2$, window size $m = 2$, maximum value $s = 2$

Profile													
	0	0	0	0	0	2	0	2	1	2	1	0	1
dual	0	2	3	1	2	3	1	2	3	1	2	3	1
	0	0	0	0	2	2	2	3	3	3	1	1	1

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Sequence of vectors from $\{0, 1, \ldots, s\}^\ell$
window of size m
No two windows have the same vector sum

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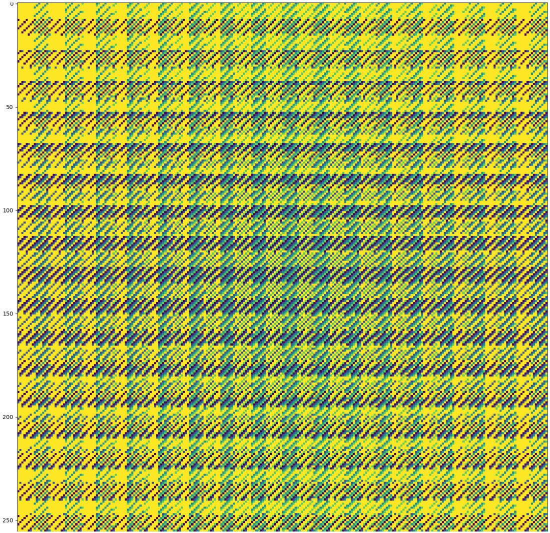
Example. vector size $\ell = 2$, window size $m = 2$, maximum value $s = 2$

Profile	0	0	2	1	0	2	1	0	2	1	0	2	1
	0	0	0	0	0	2	0	2	1	2	1	0	1
dual	0	2	3	1	2	3	1	2	3	1	2	3	1
	0	0	0	0	2	2	2	3	3	3	1	1	1

Example



size $n = 256$, colors $k = 5$, window size $m = 8$

Future work

Disk window instead of square

Error correction

Random coloring is inefficient (birthday paradox).
But what about Lovász Local Lemma?

Could Neural Networks learn to build torus packing for multisets?

Thank you!