Tournament solutions

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Reading Group Network Theory

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Introduction

A club of tennis players.
All of them played **once** against one another.

Question: who are the "best" players?

Applications:

- Sports,
- Voting and aggregation of preferences,
- Individual non-transitive preference (psychology, marketing),
- Multicriteria decision (economics, social choice).

References

- Laslier, Jean-François. Tournament solutions and majority voting. Springer, 1997.
- Brandt, Felix, Markus Brill and Paul Harrenstein. Tournament Solution. In: Brandt, Felix, Vincent Conitzer, Ulle Endriss, et al (ed.). Handbook of computational social choice. Cambridge University Press, 2016.

Plan

Basic notions

Miscellaneous solutions

“Game-theory” solutions

Conclusion

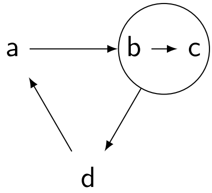
Plan

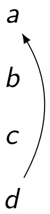
Tournament

A *tournament* T is given by:

- A (finite) list of candidates,
- A complete, antisymmetric and irreflexive relation on them.

	a	b	c	d
a		1	1	
b			1	1
c				1
d	1			





Tournament solution

A *tournament solution* S is a function whose input is a tournament and output is a non-empty subtournament. Requirements:

- Neutral (respects symmetries),
- Selects only the Condorcet winner when there is one.

	a	b	c	d
a		1	1	
b			1	1
c				1
d	1			

$\xrightarrow{S}$

	a	b	d
a		1	
b			1
d	1		

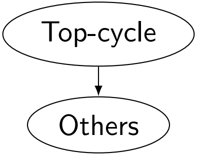
$S(T) = T_{\{a,b,d\}}$.

Loosely written:
$S(T) = \{a, b, d\}$.

Remark: all tournament solutions give the same output when there are 1, 2 or 3 candidates.

A solution: the Top-Cycle TC

The *top-cycle* $TC(T)$ is the smallest subset A of candidates such that any member of A defeats any member outside A.

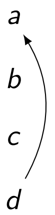


The top-cycle:

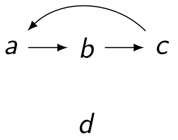
- Is neutral,
- Selects only the Condorcet winner when there is one.

$\Rightarrow$ It is a tournament solution.

Top-Cycle: examples



$$TC(T) = \{a, b, c, d\}$$



$$TC(T) = \{a, b, c\}$$

Usually, the top-cycle is quite "big". All tournament solutions that we will study are included in the top-cycle.

Monotonicity

A solution is *monotonous* iff, whenever a winner is reinforced, it does not become a loser.

Let T where x is a winner. Let T' the same as T, except for one match, which was a defeat for x in T and is a victory for x in T'. Then x must be a winner in T'.

Sum-up of properties

Monotone

Independence of the (matches between) losers

A solution is *independent of the losers* iff changing the result of a match between two losers never changes the set of winners.

Let T where x and y are losers. Let T' the same as T, except for the match between x and y. Then $S(T) = S(T')$.

In other words: the set of winners depend only on the matches between two winners and the matches between a winner and a loser.

Sum-up of properties

Monotone

Indep. of losers

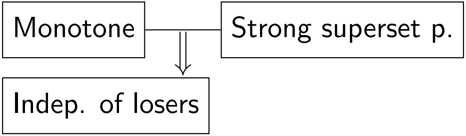
Strong Superset Property (SSP)

A solution satisfies the *Strong Superset Property* iff one does not change the set of winners by deleting some or all of the losers.

If x is a loser, then $S(T - x) = S(T)$.

If S is monotonous and verifies SSP, then S is independent of the losers.

Sum-up of properties



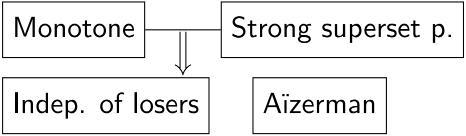
Aïzerman Property

A solution satisfies *Aïzerman property* iff when x is a loser, we have $S(T - x) \subseteq S(T)$.

Remark: later, we will see an example giving an intuitive justification of this property.

Clearly weaker than SSP, which requires $S(T - x) = S(T)$.

Sum-up of properties



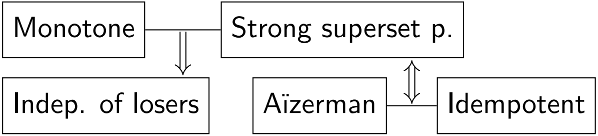
Idempotency

A solution S is *idempotent* iff $S \circ S = S$.

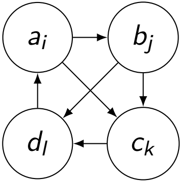
In other words: $S(T - \text{all losers}) = S(T)$.

SSP is equivalent to the conjonction of Aïzerman property and idempotency.

Sum-up of properties



Composition-Consistency

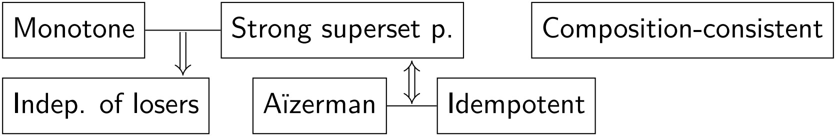


This tournament T is "decomposable":

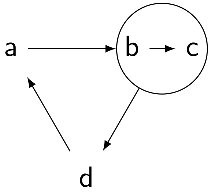
- A great tournament T' between projects a, b, c and d;
- Subtournaments T_a between variants a_i, etc.

Composition-consistency: for example, if $S(T') = \{a, b\}$, then we should have $S(T) = S(T_a) \cup S(T_b)$.

Sum-up of properties

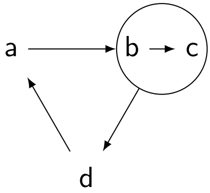


The Top-Cycle is not composition-consistent



Winner of $\{b, c\}$: b.
Winners of a 3-cycle: all candidates.
If composition-consistent, $TC(T)$ should be $\{a, b, d\}$: not true!

The Top-Cycle is not composition-consistent



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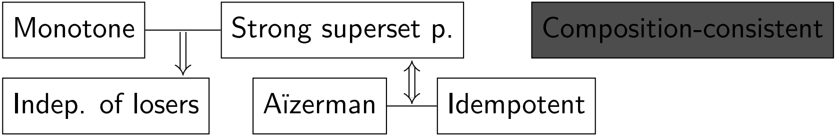
Properties of the Top-Cycle TC



Comparison of solutions

One goal: select "few" candidates.

When comparing two solutions S and S', we may have the following relation.

- $S \subseteq S'$ means $\forall T, S(T) \subseteq S'(T)$ (S is *finer* that S).

Ideally, we would like a solution with good properties and that is as fine as possible.

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Copeland solution C

Winners: candidates with most victories.

	a	b	c	d	Copeland score
a		1	1		2
b			1	1	2
c				1	1
d	1				1

$C(T) = \{a, b\}$.

Copeland solution C

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	a	b	c	d	Copeland score
a		1	1		2
b			1	1	2
c				1	1
d	1				1

$C(T) = \{a, b\}$.

A justifying model:

- There is a "true" relation of strength between candidates.
- In this true relation, there is a Condorcet winner (but the relation between other candidates may not be transitive).
- Each match is an independent observation: we get a false result with probability $p < \frac{1}{2}$.

Then Copeland winners are the maximum likelihood solutions.

Copeland solution: properties

	a	b	c	d	Copeland score
a		1	1		2
b			1	1	2
c				1	1
d	1				1

$C(T) = \{a, b\}.$

Copeland solution: properties

	a	b	c	d	Copeland score
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$$C(T) = \{a, b\}.$$

Copeland solution is monotone (obviously).

Copeland solution: properties

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$C(T) = \{a, b\}$.

Copeland solution is monotone (obviously).

If d defeated c, then we would have $C(T') = \{a, b, d\}$.
$\Rightarrow$ Not independent of losers.

Copeland solution: properties

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$\Rightarrow$ Not independent of losers.

$C(T - c) = \{a, b, d\}$.
$\Rightarrow$ Not SSP (should $= C(T)$).
$\Rightarrow$ Not even Aïzerman (should $\subseteq C(T)$).

Copeland solution: properties

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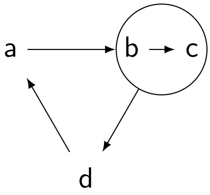
$C(T - c) = \{a, b, d\}$.
$\Rightarrow$ Not SSP (should $= C(T)$).
$\Rightarrow$ Not even Aïzerman (should $\subseteq C(T)$).

$C(C(T)) = \{a\}$.
$\Rightarrow$ Not idempotent.

Copeland solution: properties

	a	b	c	d	Copeland score
a		1	1		2
b			1	1	2
c				1	1
d	1				1



Copeland solution: properties

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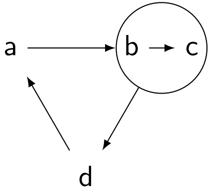

Winner of $\{b, c\}$: b.
Winners of a 3-cycle: all candidates.
If composition-consistent, $C(T)$ should be $\{a, b, d\}$: not true!

Copeland solution: properties

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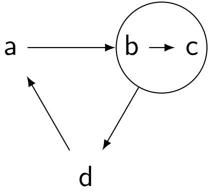

Winner of $\{b, c\}$: b.
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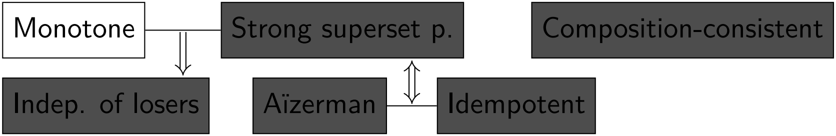
Properties of Copeland solution



Slater solution *Sl*

Model:

- There is a "true" relation of strength between candidates.
- This true relation is **transitive** (total order).
- Each match is an independent observation: we get a false result with probability $p < \frac{1}{2}$.

Find all orders that maximize the likelihood. I.e.: find all permutations that minimize the numbers of 1's under the main diagonal of the matrix.

A Slater winner is the top element of such an order.

Slater solution: example and properties

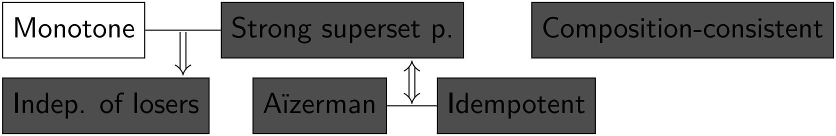
Example

	a	b	c	d
a		1	1	
b			1	1
c				1
d	1			

Only suitable order: a, b, c, d.

$\Rightarrow Sl(T) = \{a\}$.

Properties of Slater solution



Markov solution M: "Ping-pong winners"

Model:

- Start with a candidate at random.
- Choose an opponent at random and keep the winner.
- Choose a new opponent and so on...

Noting C the diagonal matrix of Copeland scores:

$$\mathbf{p}_{t+1} = \frac{1}{m-1}(T + C)\mathbf{p}_t,$$

Markov winners: candidates who have a maximal probability after infinite time.

Markov solution: example and properties

Example

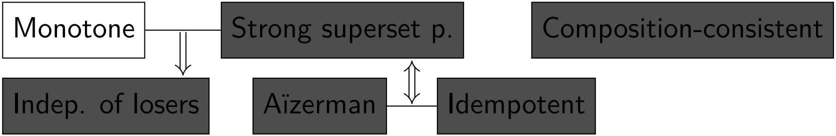
	a	b	c	d
a		1	1	
b			1	1
c				1
d	1			

$$\mathbf{p}_\infty = \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 & \\ & 2 & 1 & 1 \\ & & 1 & 1 \\ 1 & & & 1 \end{pmatrix} \mathbf{p}_\infty .$$

$$\Rightarrow \mathbf{p}_\infty = (0.4, 0.3, 0.1, 0.2).$$

$$\Rightarrow M(T) = \{a\}.$$

Properties of Markov solution



Plan

Uncovered set *UC*

A candidate x *covers* another candidate y iff x does better than y in any match: for all z, $T_{xz} \geq T_{yz}$. In particular, x must defeat y.

y is *uncovered* iff no x covers y.

	a	b	c	d
a		1	1	
b			1	1
c				1
d	1			

c is covered by b.

Other candidates are uncovered.

$\Rightarrow UC(T) = \{a, b, d\}$.

Uncovered set: link with a 2-player game

Symmetric 2-player zero-sum game: matrix $T - T^t$.

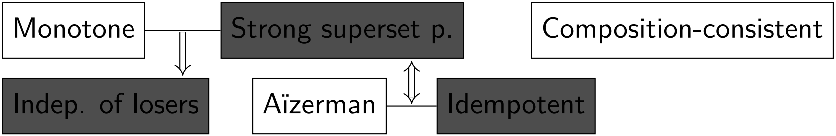
	a	b	c	d
a		1	1	-1
b	-1		1	1
c	-1	-1		1
d	1	-1	-1	

Covered candidate = dominated strategy in this game.

Uncovered set = set of undominated strategies in this game.

Uncovered set: properties

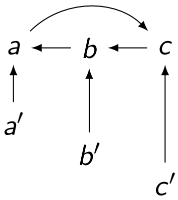
Properties of Uncovered set



Main problem: not idempotent! So...

Iterated uncovered set UC^∞

Let us go on removing covered candidates...



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

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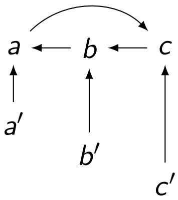

	a	b	c	a'	b'	c'
a			1		1	1
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c		1		1	1	
a'	1				1	1
b'		1				1
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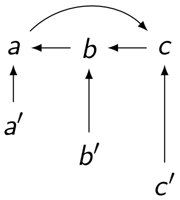
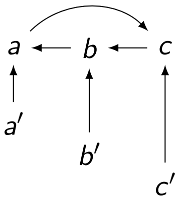


	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

$UC(T) = \{a, b, c, a', b'\}$.

Iterated uncovered set UC^∞

Let us go on removing covered candidates...



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

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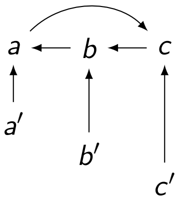
Note the example of Aïzerman property not SSP.

Iterated uncovered set UC^{∞}

Let us go on removing covered candidates...



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

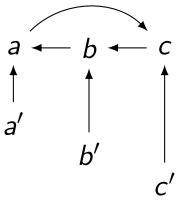
$UC(T) = \{a, b, c, a', b'\}$.

Note the example of Aïzerman property not SSP.

$UC^2(T) = \{a, b, c, a'\}$.

Iterated uncovered set UC^∞

Let us go on removing covered candidates...



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

$UC(T) = \{a, b, c, a', b'\}$.

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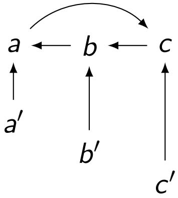
Iterated uncovered set UC^∞

Let us go on removing covered candidates...



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	1
b'		1				1
c'			1			

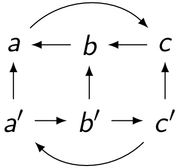
$UC(T) = \{a, b, c, a', b'\}$.

Note the example of Aïzerman property not SSP.

$UC^2(T) = \{a, b, c, a'\}$.
$UC^3(T) = UC^\infty(T) = \{a, b, c\}$.

Iterated uncovered set is not monotonous!

Another example...



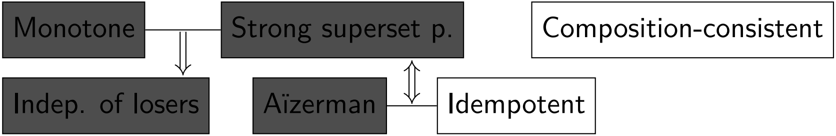
	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	
b'		1				1
c'			1	1		

$$UC(T) = UC^{\infty}(T) = \{a, b, c, a', b', c'\}.$$

If a' defeated c', then it would be the previous example and $UC^{\infty}(T') = \{a, b, c\}$: candidate a' is not a winner!
$\Rightarrow$ UC^{∞} is not monotonous.

Iterated uncovered set: properties

Properties of Iterated uncovered set



And $UC^{\infty} \subseteq UC$ (by definition).

Minimal covering set *MC*

A subset A of candidates is *covering* iff when adding another candidate $x \notin A$, then x is covered in $A \cup \{x\}$.

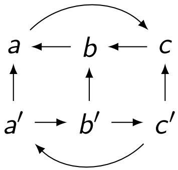
In the 2-player game, if players start to use only strategies from A, then no player has an incentive to experiment another strategy x.

Covering sets are stable by intersection $\Rightarrow$ there is a minimal covering set $MC(T)$ (and it is not empty).

In game theory: called the *weak saddle* of the game.

We have $MC \subseteq UC^{\infty}$.

Minimal covering set: example



	a	b	c	a'	b'	c'
a			1		1	1
b	1			1		1
c		1		1	1	
a'	1				1	
b'		1				1
c'			1	1		

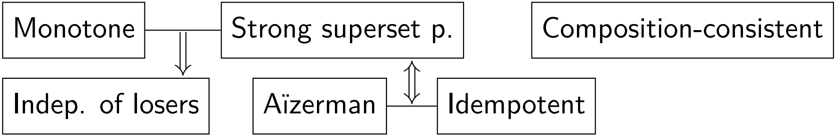
Claim: $MC(T) = \{a, b, c\}$. Proof:

- It is a covering set: if we add a' (for example), it is covered by b.
- It is minimal (since there is no Condorcet winner).

This proves that we may have $MC(T) \subsetneq UC^{\infty}(T)$.

Minimal covering set

Properties of Minimal covering set



And $MC \subseteq UC^{\infty} \subseteq UC$.

Essential set E

Lemma: the 2-player game associated to a tournament has a unique mixed-strategy equilibrium $\mathbf{p}$, and it is strict.

Essential set: all candidates who have a positive weight in $\mathbf{p}$.

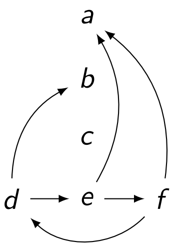
We have $E \subseteq MC$.

	a	b	c	d	Copeland score
a		1	1		2
b			1	1	2
c				1	1
d	1				1

$$\mathbf{p} = \left(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right).$$

$$\Rightarrow E(T) = \{a, b, d\}.$$

Essential set: example



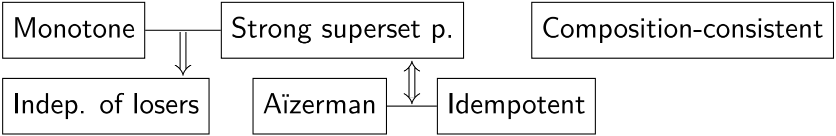
	a	b	c	d	e	f
a		1	1	1		
b			1		1	1
c				1	1	1
d		1			1	
e	1					1
f	1			1		

$\mathbf{p} = \frac{1}{9}(3, 1, 1, 1, 3, 0)$ (believe me).
$\Rightarrow E(T) = \{a, b, c, d, e\}$.

If we add f, then f is not covered. So, $\{a, b, c, d, e\}$ is not a covering set.
$\Rightarrow MC(T) = \{a, b, c, d, e, f\}$.
This proves that we may have $E(T) \subsetneq MC(T)$.

Essential set

Properties of Essential set



And $E \subseteq MC \subseteq UC^{\infty} \subseteq UC$.

Plan

Take-away

Tournament: complete, antisymmetric and irreflexive.

Tournament solution: selects a subset a "best" candidates.

Seeing the tournament as a **2-player game** leads to interesting tournament solutions. In particular:

- **Minimal covering set** $MC(T)$: if players start using only strategies from $MC(T)$, no one has an incentive to try another strategy;
- **Essential set** $E(T)$: support of the mixed-strategy equilibrium of the game.

Both these solutions are monotone, verify strong superset property and are composition-consistent.

The essential set is finer: $E \subseteq MC$.

Thanks for your attention

Questions?

Monotonicity and SSP $\Rightarrow$ Indep. of losers

Let (x, y) two losers in T, with yTx.

By monotonicity, if $y \in S(T_{<x,y>})$, then $y \in S(T)$, which is false. So, $y \notin S(T_{<x,y>})$.

Using SSP twice, we now have:

$$S(T) = S(T - y) = S(T_{<x,y>} - y) = S(T_{<x,y>}).$$