Stability Conditions for a Discrete-time Decentralised Medium Access Algorithm

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Introduction

- Medium Access Control (MAC) algorithms used to control access in wireless networks
- MAC protocol preventing neighbors from transmitting simultaneously (collision and loss of packets impossible)
- Maximal stability: all nodes in the network can transmit all arriving packets for all arriving processes
- Centralized algorithms : MaxWeight/α-fair algorithms are known to be maximally stable
  - Need a centralized controller to make decisions
- Decentralized algorithms : Carrier Sense Multiple Access (CSMA, used in IEEE 802.11)
  - Nodes have a random back-off time and transmit if they don't sense another transmission

Introduction

- Most results are known for saturated networks and cannot be reduced to unsaturated networks
- In practice, the processes are not monotonous
- Development of queue-based algorithms which provide maximal stability, but are very difficult to implement and lead to high delays
- Assume *Standard* CSMA:
  (a) Each node does not know its neighbors
  (b) Access procedure is the same for all nodes
  (c) The node does not access the network if its queue length is empty

Outline

Line and Circle Topologies

- N transmitter nodes in a circle or a line
- $\mathcal{N}_i(N)$ is the neighborhood of node N:

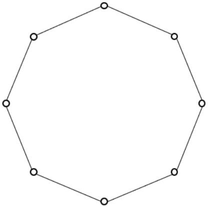
Circle topology

$$\mathcal{N}_c(i) = \begin{cases} \{N,2\} \text{ for } i=1 \\ \{N-1,1\} \text{ for } i=N \\ \{i-1,i+1\} \text{ else} \end{cases}$$

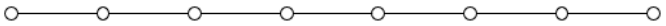
Line topology

$$\mathcal{N}_l(i) = \begin{cases} \{2\} \text{ for } i=1 \\ \{N-1\} \text{ for } i=N \\ \{i-1,i+1\} \text{ else} \end{cases}$$

Circle topology



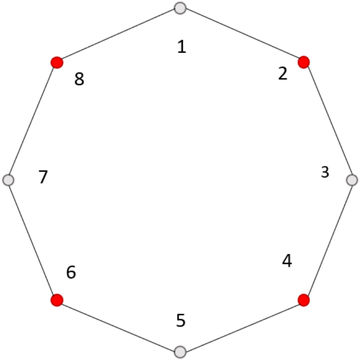
Line topology



Queuing Model

- All nodes have infinite buffer space. Time is slotted. Transmission time is equal to 1
- $Q_i(n)$: queue size at node i at time n
- $\xi_i(n)$: number of arrivals at node i at time n. $(\xi_i(n)$ are i.i.d. with $\mathbb{E}[\xi_i(n)] = \lambda$
- Transmission priorities: neighbors cannot all transmit during the same time slot (Medium Access).
  - At each time slot, priorities $\{U_1(n), \dots, U_N(n)\}$ are allocated
  - The node with priority 1 will transmit if its queue length is not zero
  - Proceed by induction: the next-highest priority node transmits if no node in its neighborhood is transmitting
  - The procedure is repeated until no node can transmit

Queuing Model



Priorities: $\{2, 1, 4, 5, 8, 3, 6, 4\}$
Transmitting nodes: $2, 4, 6, 8$
Standby nodes: $1, 3, 5, 7$

- $D_i(n)$: number of packet transmissions at queue i in time slot n
- Evolution of queue i length:

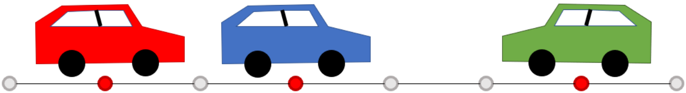
$$Q_{i+1}(n+1) = Q_i(n) - D_i(n) + \xi_i(n)$$

Outline

1. Model and Notations
2. Parking constants
3. A Loose Stability Condition
4. Towards a better stability condition

Parking constant on a line

- Transmission initiation process similar to the discrete-time parking problem



- L_k: expected number of departures in a line ok k non-empty nodes -> expected number of cars parked in a parking lot of k slots.
- $\frac{L_k}{k}$: *parking constant* (or *jamming density*)
- Known results: $\left(\frac{L_n}{n}\right)_{n\geq 3}$ is a non-increasing sequence and (see [?]):

$$L_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{2^{k-1}}{k!} (n - k + 1)$$

Lemma

$L_{k:m}$ expected number of departures from the k first nodes in a network of m non-empty nodes. Then for all $k \leq m$:

$$L_{k:m} \leq L_k$$

Proof by induction. Write:

$$L_{k:M} = \frac{1}{M}\left(\sum_{i=0}^{k-1}\left(\underbrace{1}_{\text{First node}} + \underbrace{L_{i-2} + L_{k-i-1:M-i-1}}_{\text{First } k-1 \text{ nodes}}\right) + \underbrace{L_{k-1}}_{\text{Node } k-1} + \sum_{i=k+2}^{M} \underbrace{L_{k:i-2}}_{\text{Subsequent nodes}}\right)$$

And:

$$L_k = \frac{1}{M}\left(\sum_{i=0}^{k}(1 + L_{i-2} + L_{k-i-1}) + \sum_{i=k+1}^{M} L_k\right)$$

Consequence:

$$L_{k+m} \leq L_k + L_m$$

System with reshuffling

- C_k: expected number of departures in a system of k non-empty nodes in a circle
- Introduce reshuffling (3 versions)
  - Version 1: All queues are reshuffled uniformly at random
  - Version 2: All empty queues stay where they are, all non-empty queues are reshuffled
  - Version 3: All non-empty queues are reshuffled within each non-empty segment

Theorem

For the line topology, the system with reshuffling is stable if $\lambda < \min\{L_N/N, 1/2\}$. For the circle topology, the system is stable if $\lambda < C_N/N$.

- Proof: if $\lambda < \min\{L_N/N, 1/2\}$ for the line topology, or if $\lambda < C_N/N$ for the circle topology, the average number of arrivals in any non-empty segment is lower than the number of departures

Outline

Foster-Lyapunov Theorem

Theorem (Foster, 1953)

Let X be a ϕ-irreducible discrete-time Markov chain. X is positive recurrent if and only if there exists a finite set C, a Lyapunov function L and constants $\alpha, \beta > 0$ such that:

$$\Delta V(\zeta) \equiv \mathbb{E}[L(X_1) - L(X_0) | X_0 = \zeta] \leq \beta \mathbb{1}\{\zeta \in C\} - \alpha \mathbb{1}\{\zeta \notin C\}$$

- If the state space is $\mathbb{R}^N$, it is enough to prove that there exists $K > 0$ and $\varepsilon > 0$ such that:

$$\Delta V(\zeta) < -\varepsilon$$

  whenever $|\zeta| > K$
- Idea: find a suitable function L for the queuing network and deduce a condition on λ for the system to be stable

A Loose Stability Condition

Theorem

Let ξ be such that $\xi_i(n) \stackrel{L}{\sim} \xi$. If $\lambda < 3/8$ and $\mathbb{E}[\xi^2] < \infty$, the system is stable for both topologies

- For the circle topology. Take:

$$L(x) = \sum_{i=1}^{N} (x_i + x_{i+1})^2$$

- Let $Q(0) = (Q_1(0), \dots, Q_N(0))$ be an initial condition

$$\begin{aligned}\Delta L(Q) &= \sum_{i=1}^{N} \mathbb{E}\left[(Q_i + Q_{i+1} + \xi_i + \xi_{i+1} - D_i - D_{i+1})^2 - (Q_i + Q_{i+1})^2\right] \\ &\leq \sum_{i=1}^{N} \left(\mathbb{E}\left[(\xi_i + \xi_{i+1})^2\right] + \mathbb{E}\left[(D_i + D_{i+1})^2\right]\right) + 2\sum_{i=1}^{N} (Q_i + Q_{i+1})(2\lambda - E[D_i + D_{i+1}])\end{aligned}$$

A Loose Stability Condition

- Note that:

$$\sum_{i=1}^{N} \left(\mathbb{E}\left[(\xi_i + \xi_{i+1})^2\right] + \mathbb{E}\left[(D_i + D_{i+1})^2\right] \right) \leq 2\sum_i \mathbb{E}[\xi_i^2] + 2N\lambda^2 + 4N$$

- Bound the second term:

$$\sum_{i=1}^{N} (Q_i + Q_{i+1})(2\lambda - \mathbb{E}[D_i + D_{i+1}]) = \sum_{i=1}^{N} Q_i(4\lambda - \mathbb{E}[D_{i-1}] - 2\mathbb{E}[D_i] - \mathbb{E}[D_{i+1}]])$$

- Make a case study:
  - $Q_{i-1} = Q_{i+1} = 0$: $\mathbb{E}[D_{i-1}] + 2\mathbb{E}[D_i] + \mathbb{E}[D_{i+1}] = 2$
  - $Q_{i-1} = 1$ and $Q_{i+1} = 0$: $\mathbb{E}[D_{i-1}] + 2\mathbb{E}[D_i] + \mathbb{E}[D_{i+1}] = 1 + \mathbb{E}[D_i] \geq 3/2$
  - $Q_{i-1} = Q_{i+1} = 1$:

$$\begin{aligned} \mathbb{E}[D_{i-1} + 2\mathbb{E}[D_i] + \mathbb{E}[D_{i+1}] &= (\mathbb{E}[D_{i-1} + \mathbb{E}[D_i]]) + (\mathbb{E}[D_i] + \mathbb{E}[D_{i+1}]) \\ &\geq 3/4 + 3/4 = 3/2 \end{aligned}$$

A Loose Stability Condition

- We combine the estimates:

$$\Delta L(Q) \le \underbrace{\sum_i \mathbb{E}[\xi_i^2] + 2N\lambda^2 + 4N}_{<\infty} + 2(4\lambda - 3/2)\sum_{i=1}^{N} Q_i$$

$$\le C + (8\lambda - 3)\sum_{i=1}^{N} Q_i$$

- We have $\Delta L(Q) < -\varepsilon$ if $\sum_{i=1}^{N} Q_i \ge K$ with:

$$K = \frac{C+\varepsilon}{3-8\lambda} \quad \text{and} \quad \lambda < \frac{3}{8}$$

A Loose Stability Condition

- For the line topology, we use:

$$\hat{L}(x) = \sum_{i=1}^{N-1} (x_i + x_{i+1})^2$$

- We bound the drift:

$$\Delta\hat{L}(Q) = \sum_{i=2}^{N-1} Q_i(4\lambda - \mathbb{E}[D_{i-1}] - 2\mathbb{E}[D_i] - \mathbb{E}[D_{i+1}]]) + Q_1(2\lambda \underbrace{-\mathbb{E}[D_1] - \mathbb{E}[D_2]}_{=-1}) + Q_N(2\lambda \underbrace{-\mathbb{E}[D_{N-1}] - \mathbb{E}[D_N]}_{=-1})$$

- Using the same arguments, the system is stable if $\lambda < \frac{3}{8}$

A Loose Stability Condition

- Probability of transmission of the node $2/N-1$ in the line:
  - $N=4$: $3/8$
  - $N=5$: $11/30$
- We can prove that:

$$\lim_{n\to\infty} \mathbb{P}[\text{Transmission of node 2}] = 1-e^{-1} \approx 0.3679$$

- Very well know results in Markov jump processes: the system is stable if $\lambda < \nu$

Is the condition $\lambda < \frac{3}{8}$ tight ?

Outline

Fluid limits

- Introduced by Rybko and Stolyar in [?]. Idea: study the average over large jumps in the state space
- Sequence of processes $Q^r(\cdot)$ such that $|Q^r(0)| = r$ is fixed.
- Goal: study the behavior of

$$\bar{q}(t) = \lim_{r\to\infty} \frac{1}{r}\mathbb{E}\left[|Q^r(r\,t)|\right] \tag{1}$$

Theorem (Dai, 1995, [?])

If the fluid limit model for a fixed queuing discipline is stable, i.e. there exists $T > 0$ such that $\bar{q}(T) = 0$, then the Markov chain X describing the dynamics of the network is positive Harris recurrent.

- *Remark*: the reciprocal is not true, fluid systems can be unstable and the underlying Markov chain, stable

Fluid limits

- Change the representation of the queueing network:
  - $Q_i^r(t) = Q_i^r(\lfloor t \rfloor)$ is the queue length at node i
  - $F_i^r(t) = \sum_{1 \le n \le \lfloor t \rfloor} \xi_i(n)$ is the total number of arrivals at node i up to time t
  - $H_i^r(t)$ is the total numbers of departures from node i up to time t
- Queue size at node i at time t:

$$Q_i^r(t) = Q_i^r(0) + F_i^r(t) - H_i^r(t)$$

- s: occupancy state at node i at time t, u: ranking realization (assignation of priorities)
- $d = \phi(s, u)$: transmission realization.
- Define $\Theta = \{(s, u)\}$ and $\Psi = \{(s, u, d)\}$
- Probability distribution on Θ : $\mathbb{P}_s(u, d) = \frac{1}{N!} \mathbb{1}\{d = \phi(s, u)\}$
- $G_B^r(t) = \sum_{1 \le i \le \lfloor t \rfloor} \mathbb{1}\{(s, u, d) \in B\}$: number of time slots during which event $B \in \Psi$ happened

Fluid Scaled Process

Definition

A *fluid limit* is a collection of *deterministic functions* $\chi = [(q_i, f_i, h_i)_{1\le i\le N}, (g_B)_{B\in\Psi}]$ such that there exists a subsequence r_n such that:

$$\left[\left(\frac{1}{r_n}Q_i^{r_n}(t), \frac{1}{r_n}F_i^{r_n}(t), \frac{1}{r_n}H_i^{r_n}(t)\right)_{1\le i\le N}, \left(\frac{1}{r_n}G_B^{r_n}(t)\right)_{B\in\Psi}\right] \to \chi \quad \text{u.o.c.}$$

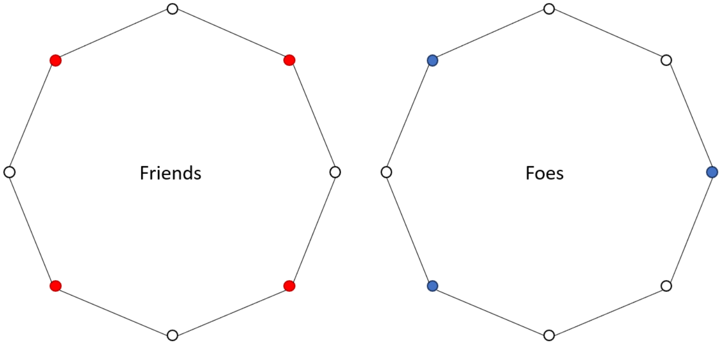
- Temporal evolution of the fluid-scaled system:

$$\bar{q}_i(t) = q_i(0) + \bar{f}_i(t) - \bar{h}_i(t) = q_i(0) + \lambda t - \bar{g}_{\{d_i=1\}}(t)$$

- Define a probability measure on Ψ:

$$\pi_t(B) = \frac{\mathrm{d}}{\mathrm{d}t} g_B(t)$$

Friends And Foes



- Nodes that are mutual friends have a higher probability of transmission, nodes that are mutual foes have a lower probability of transmission
- Edge nodes have a higher probability of transmission

Stability Condition on a Circle

- Goal: find $\varepsilon > 0$ such that, for any regular point t such that $\sum_{i=1}^{N} \bar{q}_i(t) > 0$:

$$\sum_{i=1}^{N} \bar{q}_i'(t) \leq -\varepsilon$$

- If for all $i > 0$, $\bar{q}_i(t) > 0$:

$$\bar{q}_i'(t) = \lambda - \pi_t(\{d_i = 1\}) = \lambda - \frac{C_N}{N}$$

- (C_n/n) is a non-increasing sequence for even values of n, and non-decreasing for odd values of n
- For even values $n \geq 4$, $C_n/n \geq \lim_{n\to} C_n/n = 1/2(1 - e^{-2}) > 2/5$ and for odd values $n \geq 5$, $C_n/n > C_5/5 = 2/5$

Stability Condition on a Circle

- If there is at least one i such that $\bar{q}_i(t) = 0$
- Reduce the analysis to *positive groups* of size l: groups of nodes such that $\bar{q}_{k+1}, \dots \bar{q}_{k+l}$ such that $\bar{q}_k(t) = \bar{q}_{k+l+1}(t) = 0$ and $\bar{q}_{k+i}(t) > 0$.
- We prove that for any positive group of size l, $\sum_{i=1}^{l} \bar{q}'_{k+i}(t) < -\varepsilon(l) < 0$
- Make a case study depending on the size of l:
  - If $l = 1$, we get $\bar{q}'_{k+1}(t) < \lambda + 1/2 + \lambda/4$
  - If $l = 2$, we get $\bar{q}'_{k+1}(t) + \bar{q}'_{k+2}(t) < 5\lambda/2 - 1$
  - The same goes for $l = 3$
  - If $l \geq 4$, the worst occupancy state occurs in a segment of length 7 where the middle node transmits, with probability is $179/420 > 2/5$
- We thus have:

$$\lambda < 2/5 \implies \text{The system is stable}$$

Stability Condition on a Circle

- Remind that $C_N = 1 + L_{N-3}$, and thus, $L_N / N > 2/5$
- If for all i, $q_i(t) > 0$:

$$\sum_{i=1}^{N} \bar{q}_i'(t) = N\lambda - L_N$$

  Which is negative if $\lambda < \frac{2}{5}$
- Else: case study as before
- We have to take into account border nodes
- The same result holds:

$$\lambda < 2/5 \quad \Longrightarrow \quad \text{The system is stable}$$

Conclusion

For the circle topology:

- If $\lambda > C_N/N > 2/5$, the system is always instable
- For $N = 5$, $C_N/N = 2/5$ and the bound is tight, and $\lim_{N\to\infty} C_N/N = 1/2(1 - e^{-2}) \approx 0.4323$
- Stability if $\lambda < C_N/N$ is still an open question

For the line topology:

- Some nodes receive a throughput less than 2/5
- Not an intuitive result: need to look at the overall topology and not only node throughput

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