Belief Propagation in Bayesian Networks

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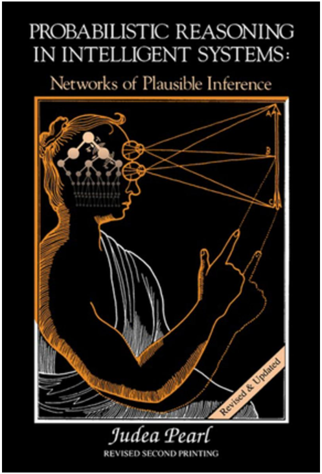
Reading Group "Network Theory"
November 5, 2018

Introduction

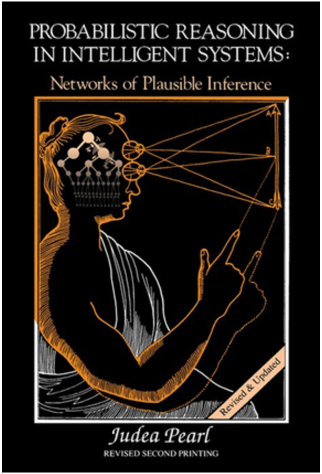
(First-order) logic
Represent causal relations between variables by a directed acyclic graph

Probabilities
Weight these causal relations by probabilities that implicitly account for non-represented variables



Introduction

"Belief networks are directed acyclic graphs in which the nodes represent propositions (or variables), the arcs signify direct dependencies between the linked propositions, and the strengths of these dependencies are quantified by conditional probabilities" (Pearl, 1986)



Bayesian networks are also ...

- A memory-efficient way of storing a PMF
- Based on simple probability rules
  (more details in a few slides)
- Inspired by human causal reasoning (Pearl, 1986, 1988)
- Used for decision taking if a utility function is provided
- Applied in many fields: medecine diagnoses, turbo-codes, (programming) language detection, ...
- Related to other models: Markov random fields, Markov chains, hidden Markov models, ...

References: Pearl's articles and book

- J. Pearl (1982). "Reverend Bayes on Inference Engines: A Distributed Hierarchical Approach". In: AAAI'82
  - → Belief propagation in causal trees
- J. Pearl (1986). "Fusion, propagation, and structuring in belief networks". In: Artificial Intelligence
  - → Belief propagation in causal trees and polytrees
- J. Pearl (1988). Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. Morgan Kaufmann
  - → A complete reference
    (thanks Achille for providing me with this book)

References: Textbooks

- T. D. Nielsen and F. V. Jensen (2007). Bayesian Networks and Decision Graphs. Springer-Verlag
  → A lot of examples in Chapters 2 and 3
- D. Koller, N. Friedman, and F. Bach (2009). Probabilistic Graphical Models: Principles and Techniques. MIT Press
- M. Jordan (Last modified in 2015). An Introduction to Probabilistic Graphical Models.
  → Definition and belief probagation
  (thanks Nathan for pointing this reference)

Outline

Reminders on probability theory

Bayesian networks

Belief propagation in trees

Belief propagation in polytrees

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Independence and conditional independence

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- A and B are **marginally independent** (written $A \perp\!\!\!\perp B$) if one of these three equivalent conditions is satisfied:
  - $P(A, B) = P(A)P(B)$
  - $P(A \mid B) = P(A)$
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- A and B are **conditonally independent** given C (written $A \perp\!\!\!\perp B \mid C$) if one of these three equivalent conditions is satisfied:
  - $P(A, B \mid C) = P(A \mid C)P(B \mid C)$
  - $P(A \mid B, C) = P(A \mid C)$
  - $P(B \mid A, C) = P(B \mid C)$

Useful rules

- **The chain rule of probabilities**
  If $A_1, \dots, A_n$ are random variables, we have

$$P(A_1, \dots, A_n) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1, A_2) \times \dots \times P(A_n \mid A_1, \dots, A_{n-1})$$

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- **Law of total probability**
  If A and B are two random variables,

$$P(B) = \sum_A P(B \mid A) P(A)$$

Useful rules

- **Bayes' rule**
  If A and B are two random variables,

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

We can see $P(A)$ as a **normalizing constant**: we can first compute $P(B \mid A) \propto P(A \mid B)P(B)$ for each value of B and then normalize to obtain $P(B \mid A)$ without computing $P(A)$

Glossary

- **Belief** in a random variable (**conviction** in french)
  Marginal distribution of this random variable
  (given the value of some observed variables)

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  (given the value of some observed variables)
- **Observe** a random variable
- **Evidence** (piece of evidence)
  The set of random variables that have been observed

Outline

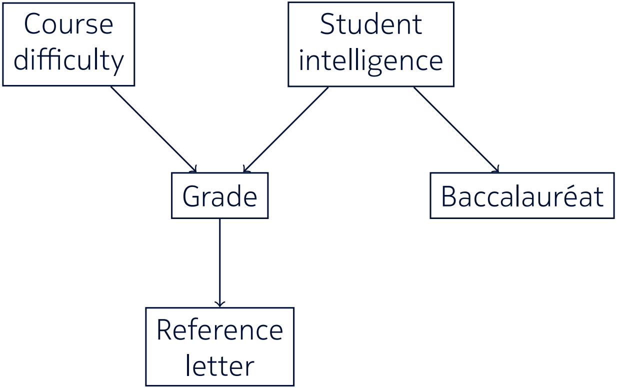
Reminders on probability theory

Bayesian networks

Belief propagation in trees

Belief propagation in polytrees

The Student example



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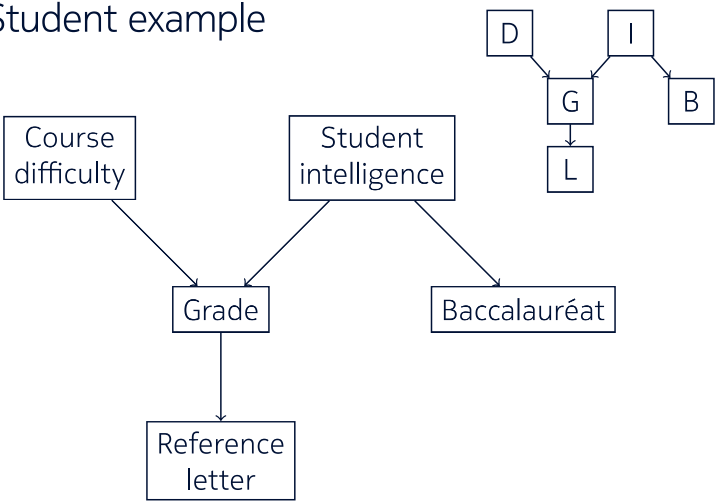

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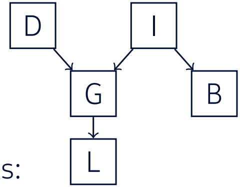

- **Local Markov property**
  Each node is conditionally independent of its non-descendants given its parents:

$$D \perp\!\!\!\perp \{I, B\}, \quad I \perp\!\!\!\perp D, \quad G \perp\!\!\!\perp B \mid \{D, I\},$$

$$B \perp\!\!\!\perp \{D, G, L\} \mid I, \quad L \perp\!\!\!\perp \{D, I, B\} \mid G$$

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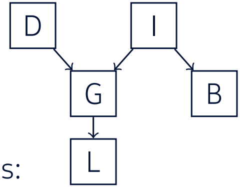

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- **Chain rule of Bayesian networks**
  By the chain rule of probabilities:

$$\begin{aligned} P(D, I, G, B, L) &= P(D)P(I \mid D)P(G \mid D, I)P(B \mid D, I, G)P(L \mid D, I, G, B), \\ &= P(D)P(I)P(G \mid D, I)P(B \mid I)P(L \mid G) \end{aligned}$$

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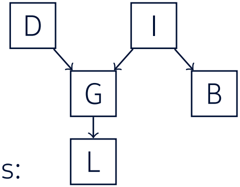

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These two definitions are equivalent

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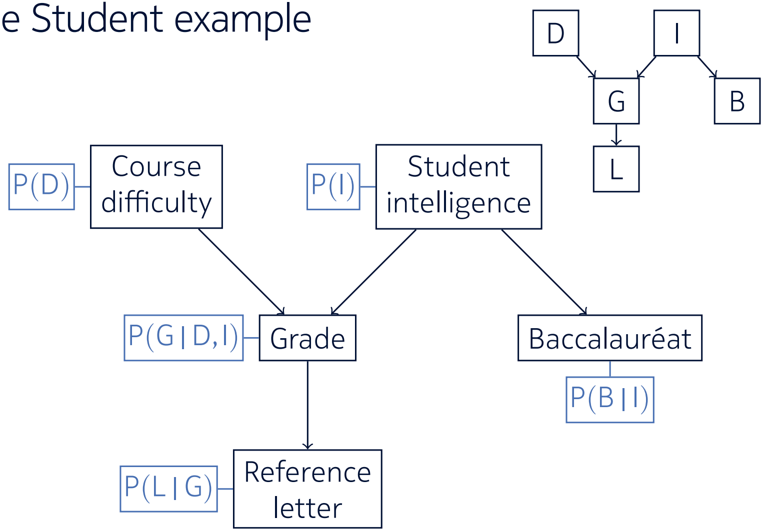
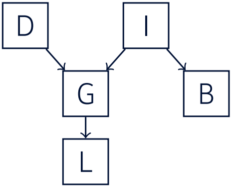


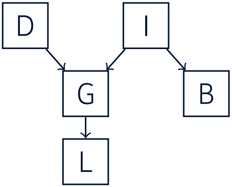
Bayesian networks in general



Described by

- A **directed acyclic graph**
  - Nodes ~ (discrete) random variables $X_1,\dots,X_n$
  - Arrows ~ conditional (in)dependencies
- Local **conditional probability tables (CPT)**
  - $P(X_i \mid parents(X_i))$ for each node X_i

Bayesian networks in general



Two equivalent definitions

- **Local Markov property**
  Each node is conditionally independent of its non-descendants given its parents
- **Chain rule of Bayesian networks**

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid \text{parents}(X_i))$$

Proof of the equivalence: Corollary 4 p.20 of (Pearl, 1988)

Base case: serial connection

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  X “causes” Y that “causes” Z

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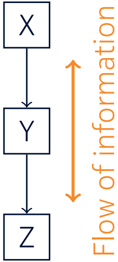

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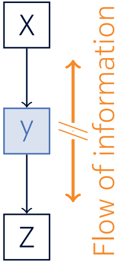

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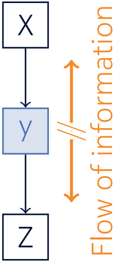

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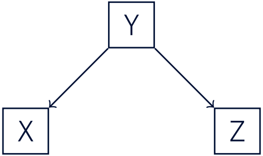
- Example: Markov chains

  Public

Base case: diverging connection



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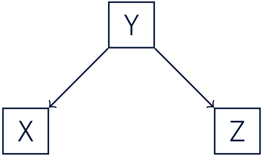

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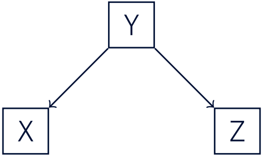

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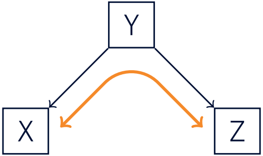

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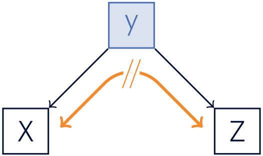

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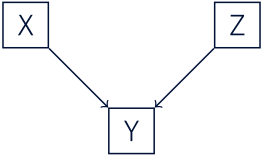

$$X \perp\!\!\!\perp Z \qquad P(X,Y,Z) = P(X)P(Y \mid X,Z)P(Z)$$

- Interpretation: two possible explanations X and Z for an observed consequence Y

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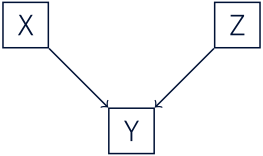
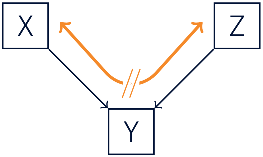
Base case: converging connection



$$X \perp\!\!\!\perp Z \qquad P(X,Y,Z) = P(X)P(Y \mid X,Z)P(Z)$$

- Interpretation: two possible explanations X and Z for an observed consequence Y
- “Explaining away” effect: information cannot flow between X and Z, unless Y is observed

Base case: converging connection



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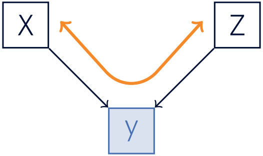

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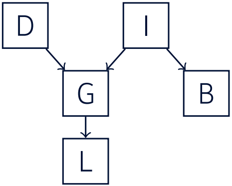
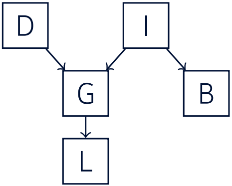
Implied independencies



Similar to the “Strong Markov property” of Markov chains

Implied independencies



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Which are correct?

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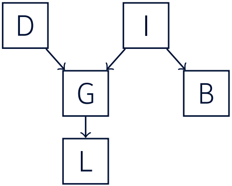

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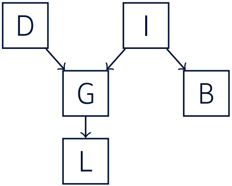

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Which are correct?

1. $G \perp\!\!\!\perp B$? No

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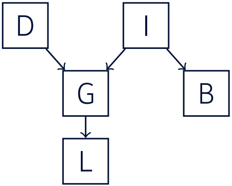

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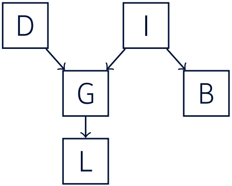

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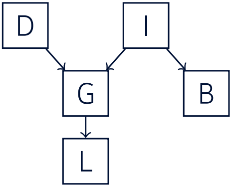

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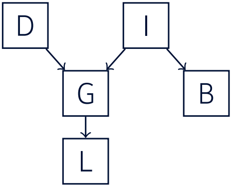

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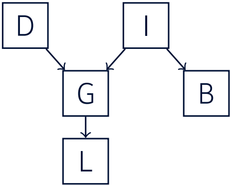

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3. $D \perp\!\!\!\perp L$? No
4. $D \perp\!\!\!\perp B$?

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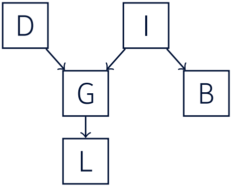
Implied independencies



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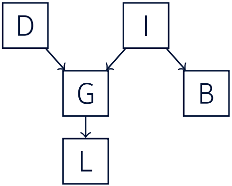
Implied independencies



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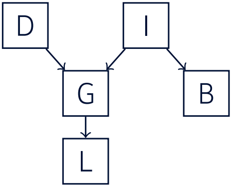

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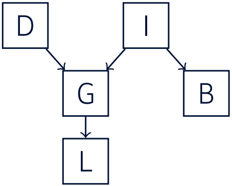
5. $D \perp\!\!\!\perp B \mid G$? No (“explaining away” effect)

Implied independencies

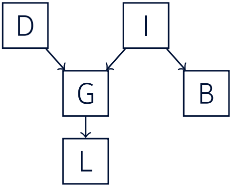


Similar to the “Strong Markov property” of Markov chains

Which are correct?

1. $G \perp\!\!\!\perp B$? No
2. $B \perp\!\!\!\perp L$? No
3. $D \perp\!\!\!\perp L$? No
4. $D \perp\!\!\!\perp B$? Yes (by the local Markov property applied to D)
5. $D \perp\!\!\!\perp B \mid G$? No (“explaining away” effect)
6. $D \perp\!\!\!\perp B \mid \{I, G\}$?

Implied independencies



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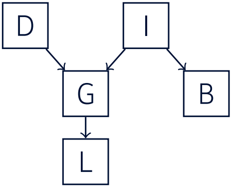

Proof of 6 $D \perp\!\!\!\perp B \mid \{I, G\}$

$$P(D, B \mid I, G)$$

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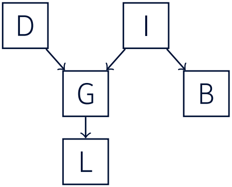
Implied independencies

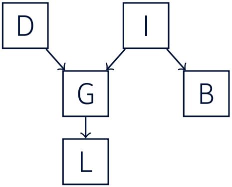


Proof of 6 $D \perp\!\!\!\perp B \mid \{I, G\}$

$$P(D, B \mid I, G)$$

$$= P(D \mid G, I) P(B \mid I, G) \qquad \square$$

Implied independencies

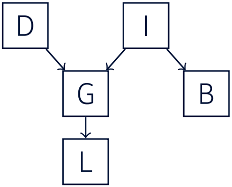


Proof of 6 $D \perp\!\!\!\perp B \mid \{I, G\}$

$$P(D, B \mid I, G) = \frac{P(G \mid D, I, B) P(D, B \mid I)}{P(G \mid I)} \quad \left(\begin{matrix}\text{Bayes'} \\ \text{rule}\end{matrix}\right)$$

$$= P(D \mid G, I) P(B \mid I, G) \qquad \square$$

Implied independencies



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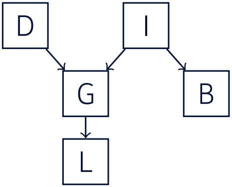
Implied independencies

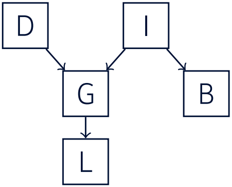


Proof of 6 $D \perp\!\!\!\perp B \mid \{I, G\}$

$$
\begin{aligned}
P(D,B \mid I,G) &= \frac{P(G \mid D,I,B)P(D,B \mid I)}{P(G \mid I)} && \begin{pmatrix}\text{Bayes'} \\ \text{rule}\end{pmatrix} \\
&= \frac{P(G \mid D,I)P(D,B \mid I)}{P(G \mid I)} && \begin{pmatrix}\text{local Markov property} \\ \text{applied to G}\end{pmatrix} \\
&= \frac{P(G \mid D,I)P(D \mid I)P(B \mid D,I)}{P(G \mid I)} && \begin{pmatrix}\text{definition of condi-} \\ \text{tional probabilities}\end{pmatrix}
\end{aligned}
$$

$$= P(D \mid G,I)P(B \mid I,G) \qquad \square$$

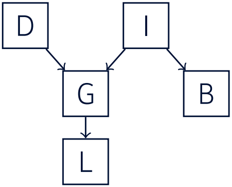
Implied independencies



Proof of (6) $D \perp\!\!\!\perp B \mid \{I, G\}$

$$
\begin{aligned}
P(D,B \mid I,G) &= \frac{P(G \mid D,I,B)P(D,B \mid I)}{P(G \mid I)} && \text{(Bayes' rule)} \\
&= \frac{P(G \mid D,I)P(D,B \mid I)}{P(G \mid I)} && \text{(local Markov property applied to G)} \\
&= \frac{P(G \mid D,I)P(D \mid I)P(B \mid D,I)}{P(G \mid I)} && \text{(definition of conditional probabilities)} \\
&= \frac{P(G \mid D,I)P(D \mid I)}{P(G \mid I)} P(B \mid D,I) \\
&= P(D \mid G,I)P(B \mid I,G) && \square
\end{aligned}
$$

Implied independencies



Proof of 6 $D \perp\!\!\!\perp B \mid \{I, G\}$

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P(D,B \mid I,G) &= \frac{P(G \mid D,I,B)P(D,B \mid I)}{P(G \mid I)} && \begin{pmatrix}\text{Bayes'}\\ \text{rule}\end{pmatrix} \\
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&= \frac{P(G \mid D,I)P(D \mid I)P(B \mid D,I)}{P(G \mid I)} && \begin{pmatrix}\text{definition of condi-}\\ \text{tional probabilities}\end{pmatrix} \\
&= \frac{P(G \mid D,I)P(D \mid I)}{P(G \mid I)} P(B \mid D,I) \\
&= P(D \mid G,I)P(B \mid D,I) && \begin{pmatrix}\text{Bayes'}\\ \text{rule}\end{pmatrix} \\
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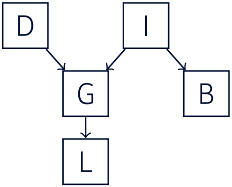

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$$

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What about the time complexity?

Inference

“A guess that you make or an opinion that you form based on the information that you have” (Cambridge dictionary)

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- Dynamic programming

Outline

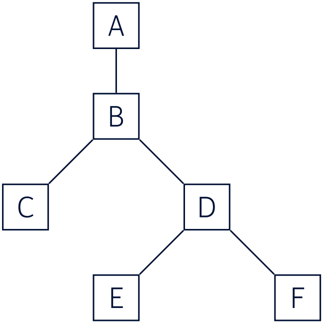
Reminders on probability theory

Bayesian networks

Belief propagation in trees

Belief propagation in polytrees

Tree Bayesian network



Each node (except the root) has at most one parent

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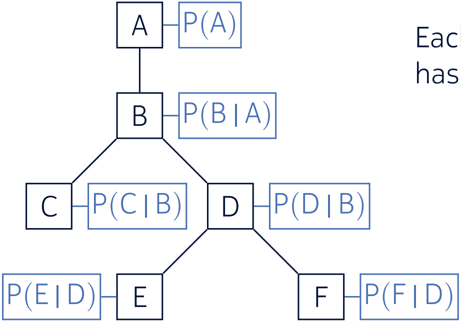
Tree Bayesian network



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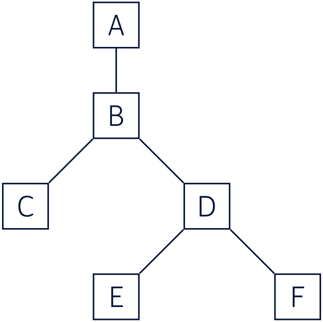

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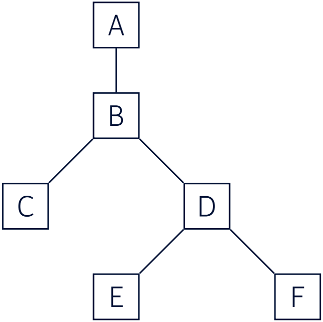

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Each node **separates** the tree: its non-descendants and the subtrees rooted at each of its children are conditionally independent given this node

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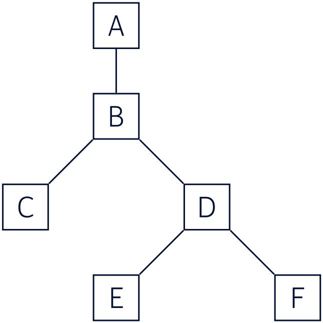

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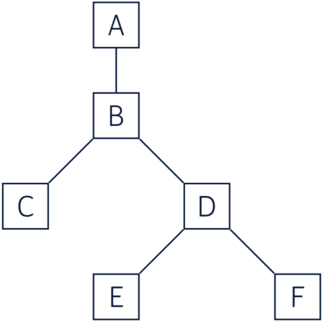
Remark: We will explain the propagation algorithm on this toy example borrowed from (Pearl, 1988)

No evidence



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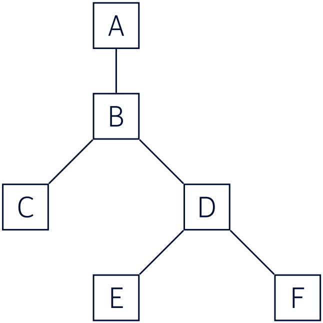
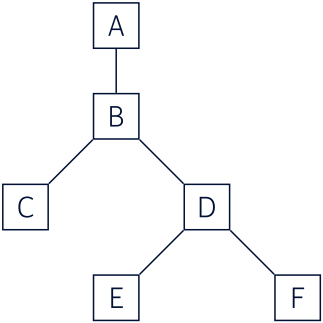


- P(A): parameter

No evidence



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- $P(B) = \sum_{A} P(B \mid A) P(A)$

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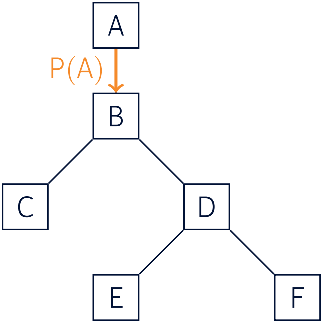
No evidence



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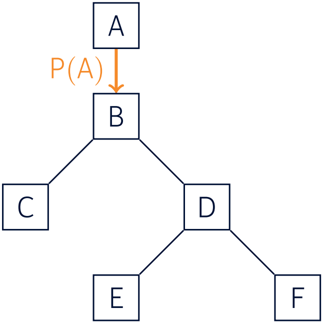

- $P(A)$: parameter
- $P(B) = \sum_{A} P(B \mid A) P(A)$
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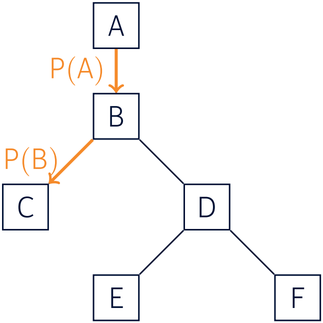

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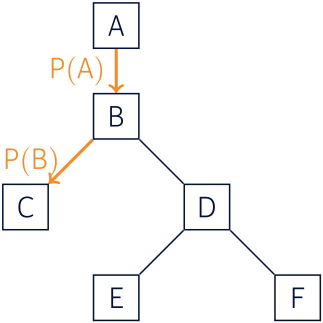

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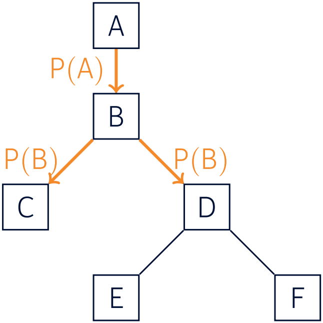

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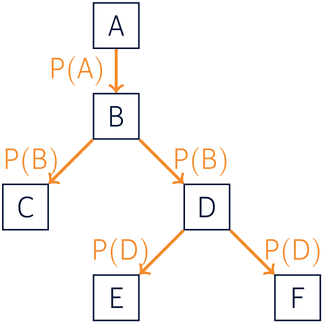

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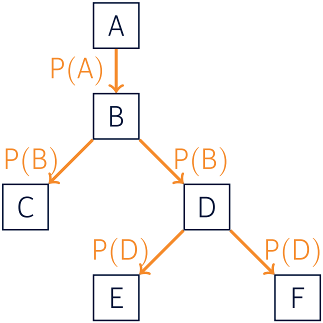

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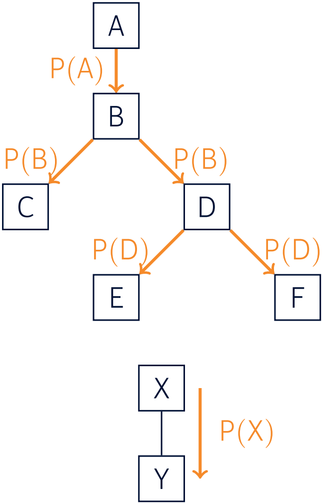
Top-down propagation
Complexity $O(nr^2)$

No evidence



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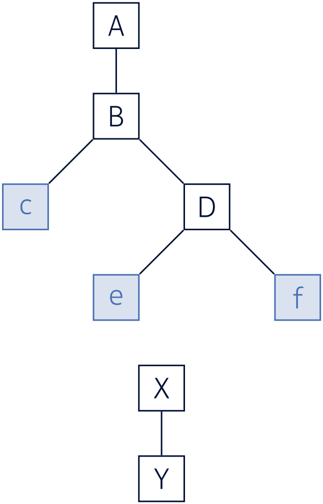
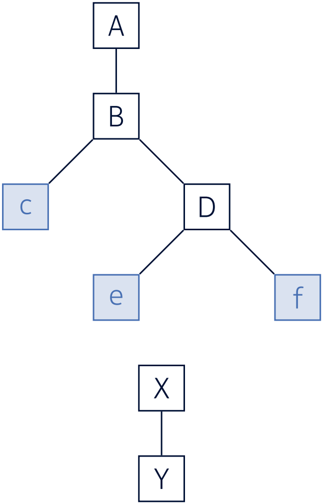
Three pieces of evidence



- **Evidence:** We observe that $C = c$, $E = e$, and $F = f$

Three pieces of evidence



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- **Objective:** Compute the belief $BEL(X) = P(X \mid c, e, f)$ of each node X

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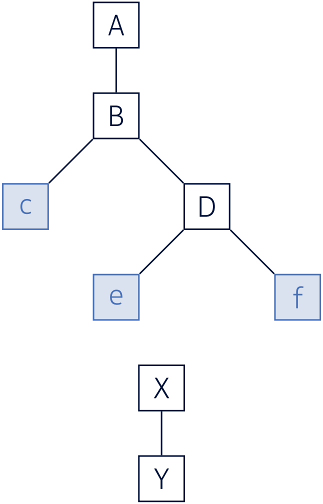

- **Evidence:** We observe that $C = c$, $E = e$, and $F = f$
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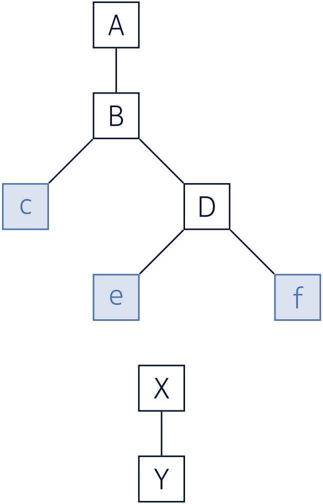
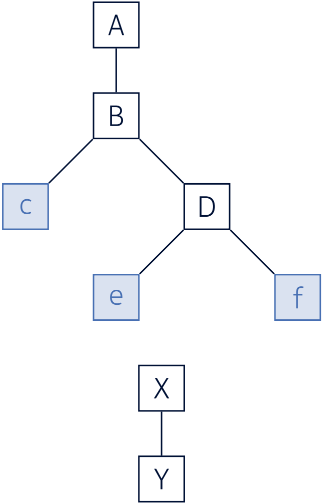
- **Principle:** Propagate the information through the network, starting from the evidence nodes

Causal and diagnostic support



By Bayes' rule:

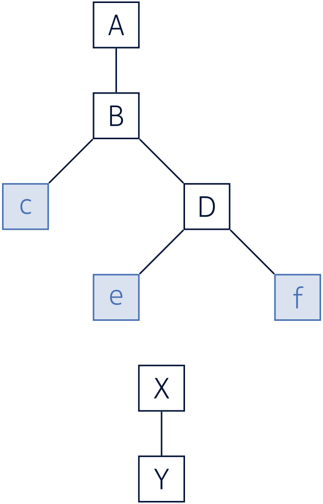
Causal and diagnostic support



By Bayes' rule:

- $P(D \mid c, e, f) = \dfrac{P(e, f \mid D, c) P(D \mid c)}{P(e, f \mid c)}$

Causal and diagnostic support



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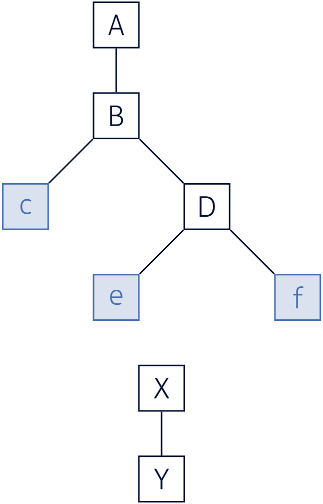
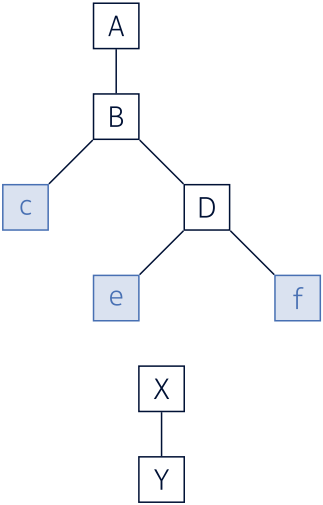
Causal and diagnostic support



By Bayes' rule:

- $$P(D \mid c,e,f) = \frac{P(e,f \mid D,c)P(D \mid c)}{P(e,f \mid c)} = \frac{P(e,f \mid D)P(D \mid c)}{P(e,f \mid c)} \propto P(e,f \mid D)P(D \mid c)$$

Causal and diagnostic support



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- $$P(B \mid c,e,f) = \frac{P(c,e,f \mid B)P(B)}{P(c,e,f)}$$
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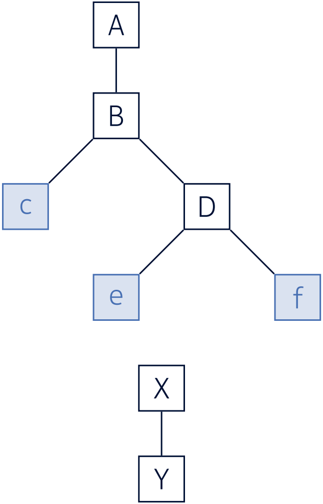
Causal and diagnostic support



By Bayes' rule:

- $$P(B \mid c,e,f) = \frac{P(c,e,f \mid B)P(B)}{P(c,e,f)} \propto P(c,e,f \mid B)P(B)$$
- $$P(D \mid c,e,f) = \frac{P(e,f \mid D,c)P(D \mid c)}{P(e,f \mid c)} = \frac{P(e,f \mid D)P(D \mid c)}{P(e,f \mid c)} \propto P(e,f \mid D)P(D \mid c)$$

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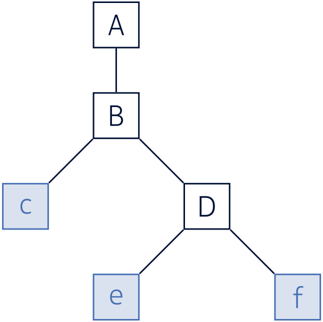
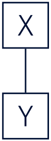
Causal and diagnostic support

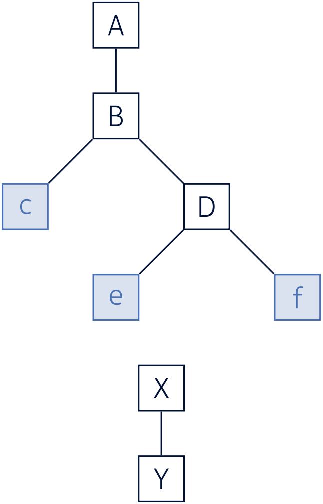




By Bayes' rule:

- $$P(A \mid c,e,f) = \frac{P(c,e,f \mid A)P(A)}{P(c,e,f)}$$
- $$P(B \mid c,e,f) = \frac{P(c,e,f \mid B)P(B)}{P(c,e,f)} \propto P(c,e,f \mid B)P(B)$$
- $$P(D \mid c,e,f) = \frac{P(e,f \mid D,c)P(D \mid c)}{P(e,f \mid c)} = \frac{P(e,f \mid D)P(D \mid c)}{P(e,f \mid c)} \propto P(e,f \mid D)P(D \mid c)$$

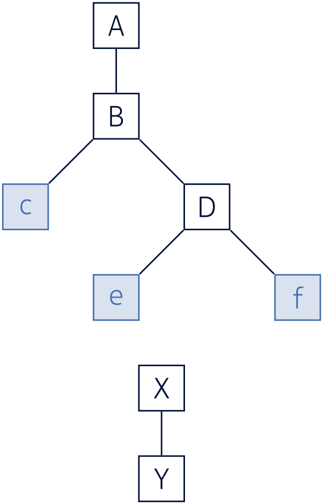
Causal and diagnostic support



By Bayes' rule:

- $$P(A \mid c,e,f) = \frac{P(c,e,f \mid A)P(A)}{P(c,e,f)} \propto P(c,e,f \mid A)P(A)$$
- $$P(B \mid c,e,f) = \frac{P(c,e,f \mid B)P(B)}{P(c,e,f)} \propto P(c,e,f \mid B)P(B)$$
- $$P(D \mid c,e,f) = \frac{P(e,f \mid D,c)P(D \mid c)}{P(e,f \mid c)} = \frac{P(e,f \mid D)P(D \mid c)}{P(e,f \mid c)} \propto P(e,f \mid D)P(D \mid c)$$

Causal and diagnostic support

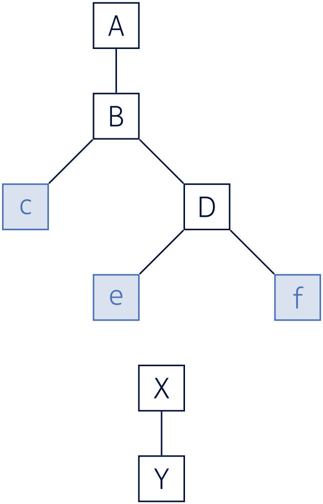


For each X, we compute

- Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below X}\end{matrix} \mid X\right)$
  Bottom-up propagation
- Causal support $P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above X}\end{matrix}\right)$
  Top-down propagation

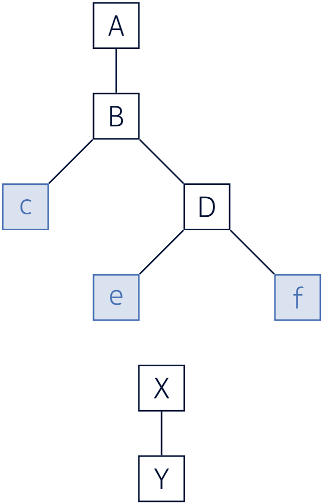
$$\mathrm{BEL}(X) \propto P\left(\begin{matrix}\text{evidence}\\ \text{below X}\end{matrix} \mid X\right) \times P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above X}\end{matrix}\right)$$

Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \mid X\right)$



Bottom-up propagation

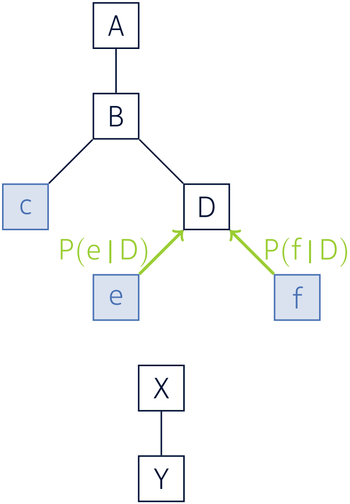
Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below X}\end{matrix} \mid X\right)$



Bottom-up propagation

- $P(e, f \mid D) = P(e \mid D) P(f \mid D)$

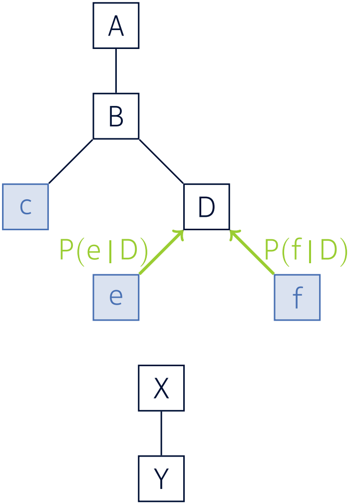
Diagnostic support $P\left(\begin{matrix}\text{evidence} \\ \text{below } X\end{matrix} \,\middle|\, X\right)$



Bottom-up propagation

- $P(e, f \mid D) = P(e \mid D) P(f \mid D)$

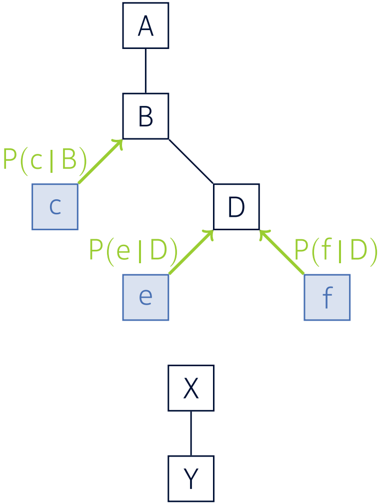
Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below X}\end{matrix} \mid X\right)$



Bottom-up propagation

- $P(e, f \mid D) = P(e \mid D) P(f \mid D)$
- $P(c, e, f \mid B) = P(c \mid B) P(e, f \mid B)$

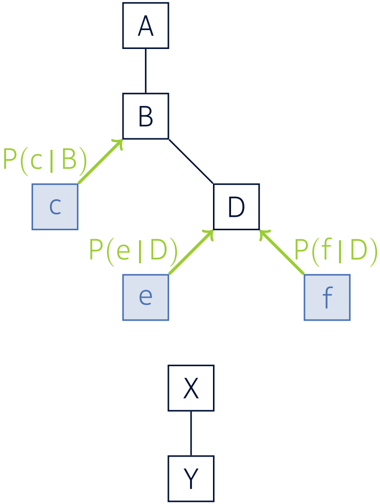
Diagnostic support $P\left(\begin{smallmatrix}\text{evidence}\\ \text{below } X\end{smallmatrix} \mid X\right)$



Bottom-up propagation

- $P(e,f \mid D) = P(e \mid D)P(f \mid D)$
- $P(c,e,f \mid B) = P(c \mid B)P(e,f \mid B)$

Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \mid X\right)$



Bottom-up propagation

- $P(e,f \mid D) = P(e \mid D)P(f \mid D)$
- $P(c,e,f \mid B) = P(c \mid B)P(e,f \mid B)$

Compute $P(e,f \mid B)$:

$$P(e,f \mid B) = \sum_D P(e,f \mid B,D)P(D \mid B)$$
$$= \sum_D P(e,f \mid D)P(D \mid B)$$

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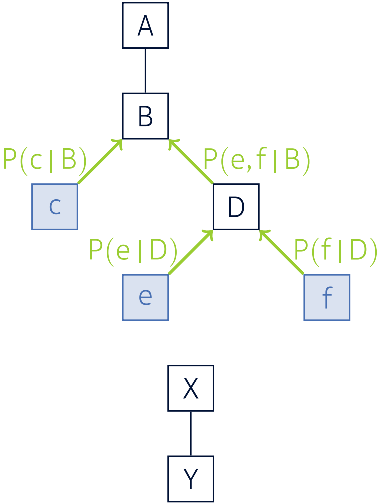
Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \mid X\right)$



Bottom-up propagation

- $P(e,f \mid D) = P(e \mid D)P(f \mid D)$
- $P(c,e,f \mid B) = P(c \mid B)P(e,f \mid B)$

Compute $P(e,f \mid B)$:

$$P(e,f \mid B) = \sum_D P(e,f \mid B,D)P(D \mid B)$$
$$= \sum_D P(e,f \mid D)P(D \mid B)$$

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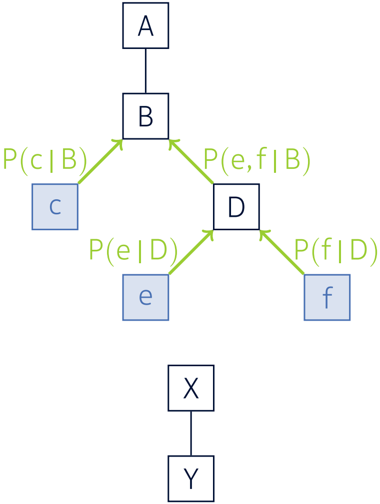
Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below X}\end{matrix} \,\middle|\, X\right)$



- $P(c,e,f \mid A) = P(c,e,f \mid A)$

  Compute $P(c,e,f \mid A)$:

  $$\begin{aligned} &P(c,e,f \mid A) \\ &= \sum_B P(c,e,f \mid A,B)P(B \mid A) \\ &= \sum_B P(c,e,f \mid B)P(B \mid A) \end{aligned}$$

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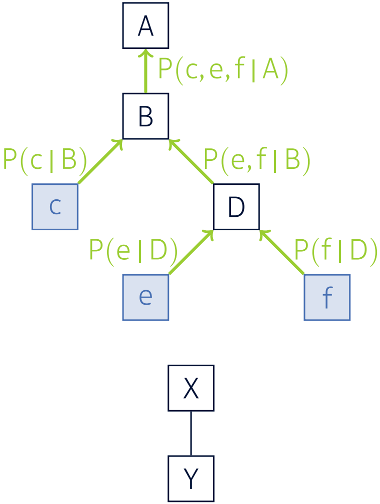
Diagnostic support $P\left(\begin{matrix}\text{evidence}\\\text{below } X\end{matrix} \mid X\right)$

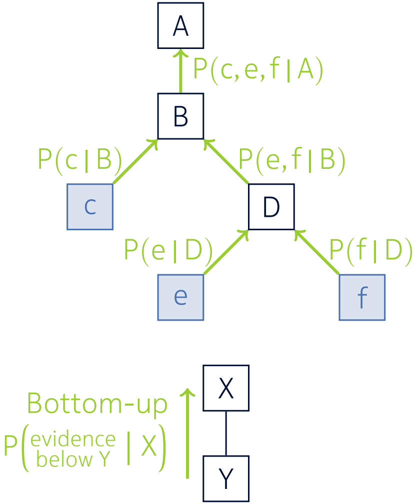


- $P(c,e,f \mid A) = P(c,e,f \mid A)$

  Compute $P(c,e,f \mid A)$:

  $$P(c,e,f \mid A) = \sum_B P(c,e,f \mid A,B)P(B \mid A) = \sum_B P(c,e,f \mid B)P(B \mid A)$$

Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \mid X\right)$



- $P(c,e,f \mid A) = P(c,e,f \mid A)$

  Compute $P(c,e,f \mid A)$:

  $$P(c,e,f \mid A)$$
  $$= \sum_B P(c,e,f \mid A,B)P(B \mid A)$$
  $$= \sum_B P(c,e,f \mid B)P(B \mid A)$$

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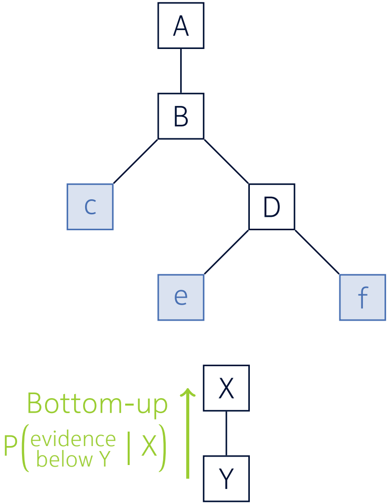
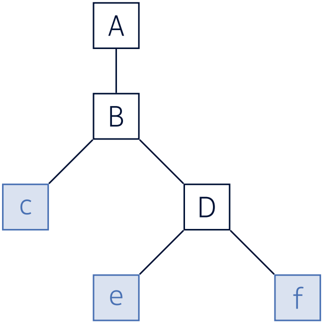
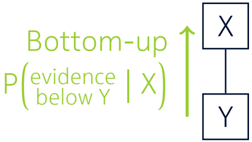
Causal support $P\left(X \mid \substack{\text{evidence} \\ \text{above } X}\right)$



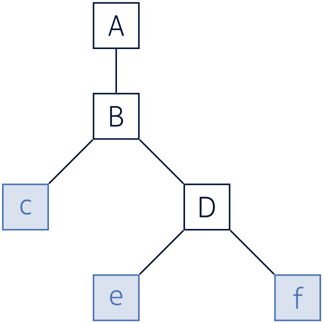
Causal support $P\left(X \mid \begin{array}{c}\text{evidence}\\ \text{above } X\end{array}\right)$





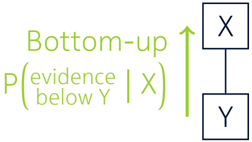
- P(A): parameter

Causal support $P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$

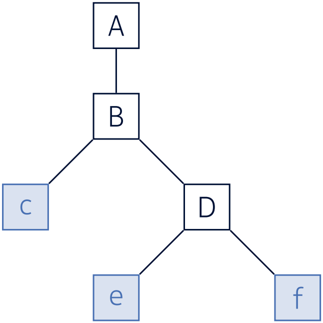


- $P(A)$: parameter

  $\rightarrow \mathrm{BEL}(A) \propto P(c,e,f \mid A)P(A)$



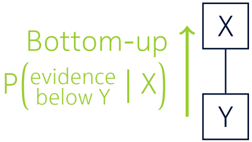
Causal support $P\left(X \mid \begin{array}{c}\text{evidence}\\ \text{above } X\end{array}\right)$



- $P(A)$: parameter

  $\rightarrow BEL(A) \propto P(c,e,f \mid A)P(A)$

- $P(B) = \sum_A P(B \mid A)P(A)$



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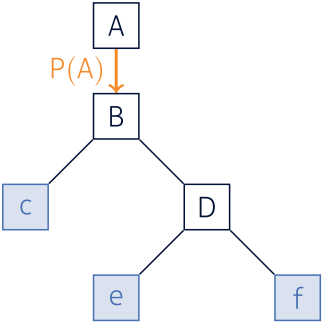
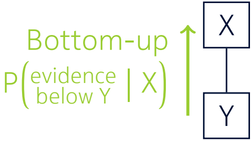
Causal support $P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$



- $P(A)$: parameter

  $\rightarrow \mathrm{BEL}(A) \propto P(c, e, f \mid A)P(A)$

- $P(B) = \sum_{A} P(B \mid A)P(A)$



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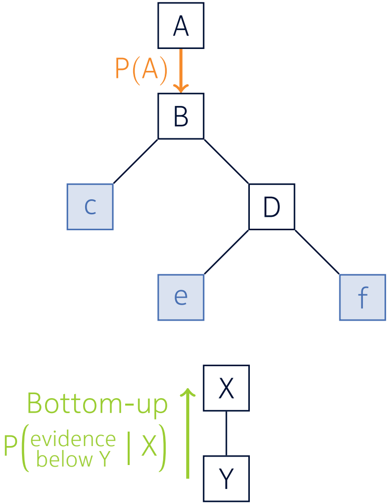
Causal support $P\left(X \mid \begin{matrix}\text{evidence} \\ \text{above } X\end{matrix}\right)$



- $P(A)$: parameter

  $\rightarrow BEL(A) \propto P(c,e,f \mid A)P(A)$

- $P(B) = \sum_A P(B \mid A)P(A)$

  $\rightarrow BEL(B) \propto P(c,e,f \mid B)P(B)$

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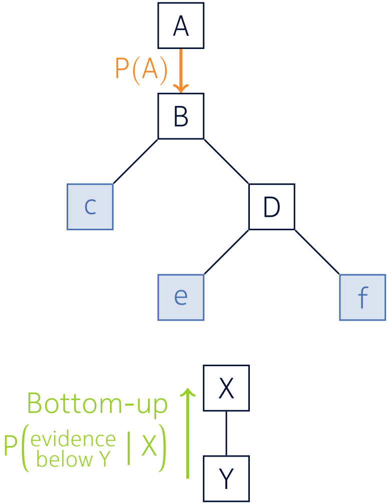
Causal support $P\left(X \mid \begin{matrix}\text{evidence} \\ \text{above } X\end{matrix}\right)$



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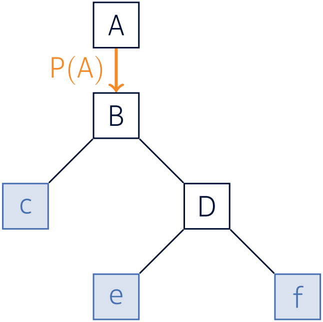
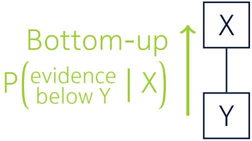
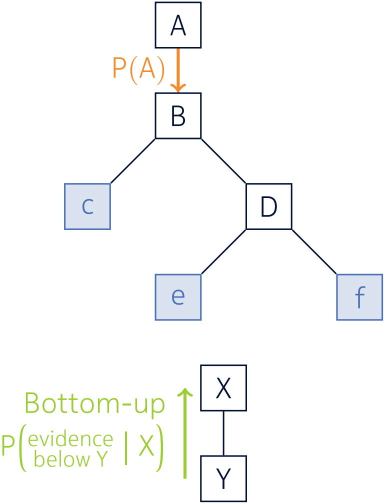
Causal support $P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$



- $P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$

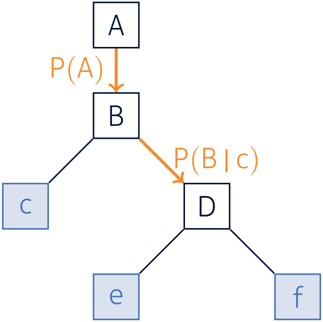


Causal support $P\left(X \mid \begin{smallmatrix}\text{evidence}\\ \text{above } X\end{smallmatrix}\right)$

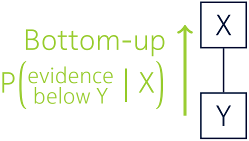


- $$P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$$
$$= \sum_B P(D \mid B) P(B \mid c)$$

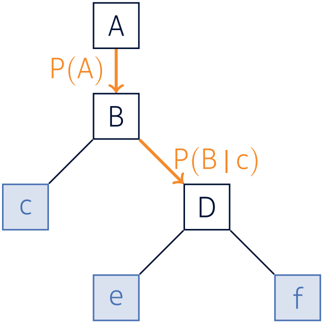
Causal support $P\left(X \mid \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$



- $P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$
  $= \sum_B P(D \mid B) P(B \mid c)$



Causal support $P\left(X \mid \begin{smallmatrix}\text{evidence}\\\text{above } X\end{smallmatrix}\right)$



- $P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$

  $= \sum_B P(D \mid B) P(B \mid c)$

Compute $P(B \mid c)$:

$$P(B \mid c, e, f) = \frac{P(e, f \mid B, c) P(B \mid c)}{P(e, f \mid c)}$$

i.e. $P(B \mid c) \propto \dfrac{BEL(B)}{P(e, f \mid B)}$

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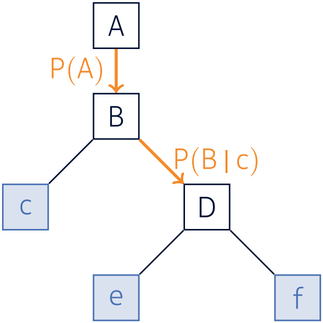
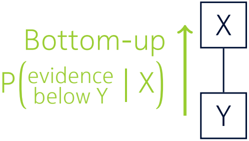
Causal support $P\left(X \,\middle|\, \substack{\text{evidence} \\ \text{above } X}\right)$





- $$P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$$
  $$= \sum_B P(D \mid B) P(B \mid c)$$

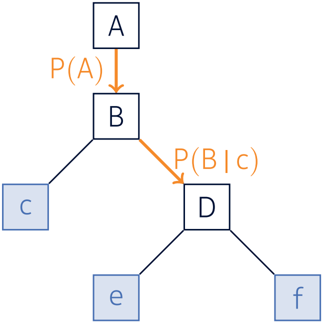
Compute $P(B \mid c)$:

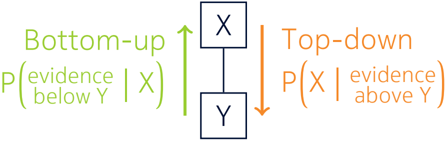
$$P(B \mid c, e, f) = \frac{P(e, f \mid B, c) P(B \mid c)}{P(e, f \mid c)}$$

i.e. $$P(B \mid c) \propto \frac{BEL(B)}{P(e, f \mid B)}$$

$\rightarrow BEL(D) \propto P(e, f \mid D) P(D \mid c)$

Causal support $P\left(X \mid \begin{smallmatrix}\text{evidence}\\ \text{above } X\end{smallmatrix}\right)$





- $$P(D \mid c) = \sum_B P(D \mid B, c) P(B \mid c)$$
$$= \sum_B P(D \mid B) P(B \mid c)$$

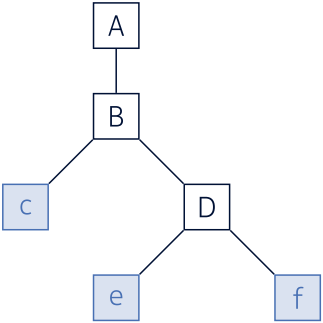
Compute $P(B \mid c)$:

$$P(B \mid c, e, f) = \frac{P(e, f \mid B, c) P(B \mid c)}{P(e, f \mid c)}$$

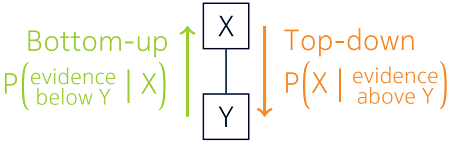
i.e. $P(B \mid c) \propto \dfrac{BEL(B)}{P(e, f \mid B)}$

$\rightarrow BEL(D) \propto P(e, f \mid D) P(D \mid c)$

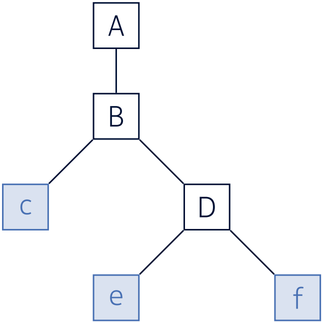
Summary

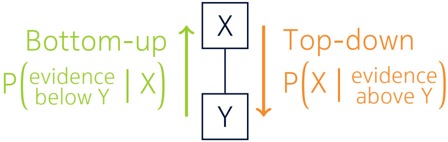


Algorithm



Summary

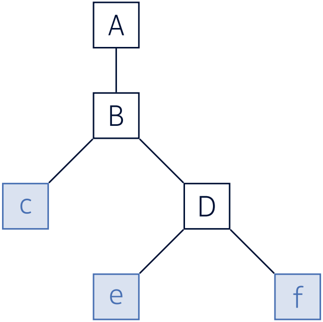




Algorithm

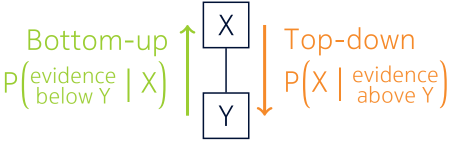
- Diagnostic support $P\left(\begin{smallmatrix}\text{evidence}\\ \text{below } X\end{smallmatrix} \mid X\right)$
  Bottom-up propagation

Summary



Algorithm

- Diagnostic support $P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \,\middle|\, X\right)$
  Bottom-up propagation
- Causal support $P\left(X \,\middle|\, \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$
  Top-down propagation



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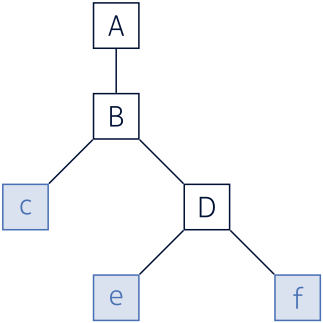
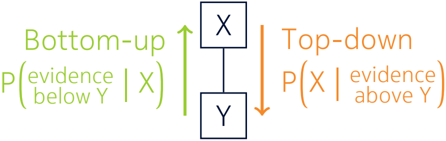
Summary



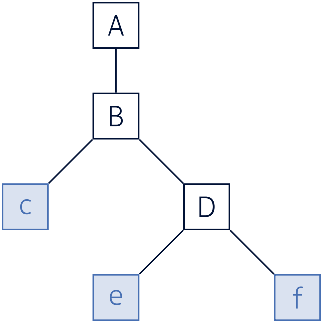
Algorithm

- Diagnostic support $P\left(\begin{smallmatrix}\text{evidence}\\ \text{below } X\end{smallmatrix} \mid X\right)$
  Bottom-up propagation
- Causal support $P\left(X \mid \begin{smallmatrix}\text{evidence}\\ \text{above } X\end{smallmatrix}\right)$
  Top-down propagation

In general



Summary



Algorithm

- Diagnostic support $P\left(\begin{smallmatrix}\text{evidence}\\ \text{below } X\end{smallmatrix} \mid X\right)$
  Bottom-up propagation
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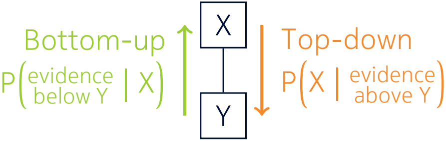
In general

- Use a topological ordering



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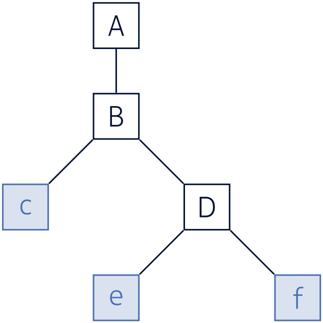

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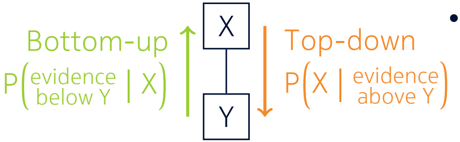
In general

- Use a topological ordering
- Complexity: $O(rd^{\downarrow} + r^2 + r)$



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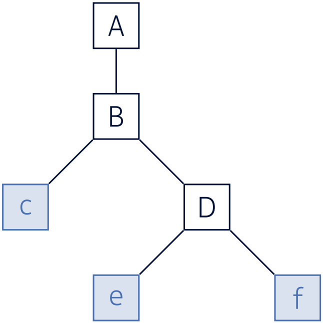
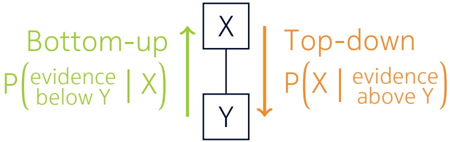
Additional remarks





- If the evidence node is not a leaf: add a phantom node

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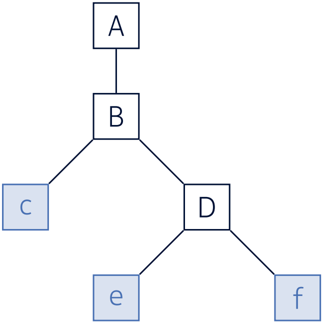
Additional remarks



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  - Belief, causal and diagnostic supports, messages ~ Vectors
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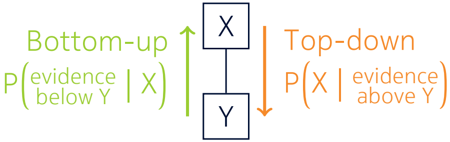

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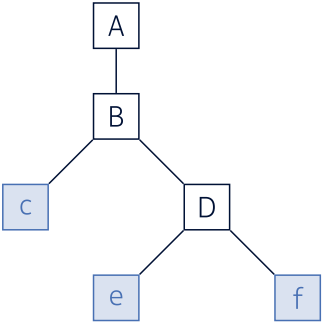
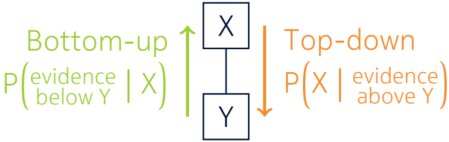




- If the evidence node is not a leaf: add a phantom node
- The calculations can be written as matrix products
  - Belief, causal and diagnostic supports, messages ~ Vectors
  - CPT ~ Matrix
- Asynchronous / parallel updates: acknowledgements (Pearl, 1982, 1986, 1988)

Outline

Reminders on probability theory

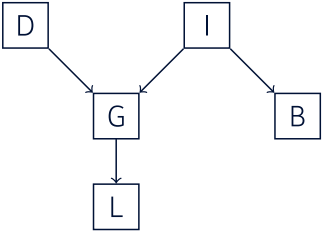
Bayesian networks

Belief propagation in trees

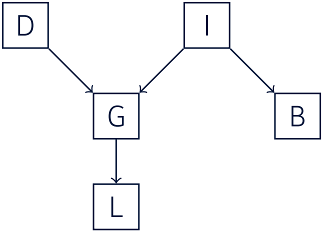
Belief propagation in polytrees

Polytree (or singly-connected) Bayesian network

The underlying undirected graph is a tree



Polytree (or singly-connected) Bayesian network



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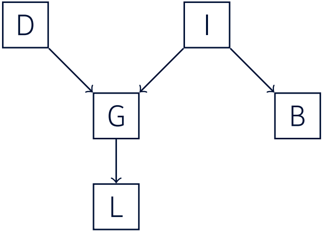
Polytree (or singly-connected) Bayesian network



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- Given a node, the non-descendants and the subtrees rooted at each child are independent

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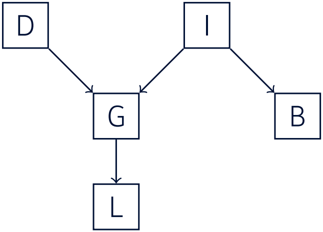
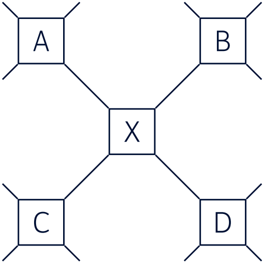


The underlying undirected graph is a tree

Separation properties

- Given a node, the non-descendants and the subtrees rooted at each child are independent
- If we don't condition on a node nor any of its descendants, the inversed subtrees rooted at its ancestors are independent

Belief propagation



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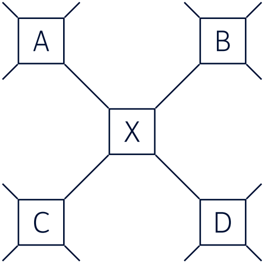
Belief propagation

$$P(X \mid \text{evidence}) \propto P\left(\begin{matrix}\text{evidence}\\ \text{below } X\end{matrix} \,\middle|\, X\right) \times P\left(X \,\middle|\, \begin{matrix}\text{evidence}\\ \text{above } X\end{matrix}\right)$$



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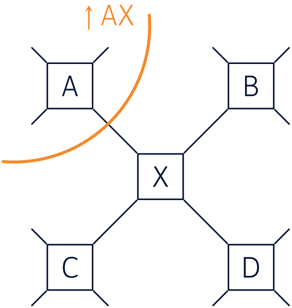

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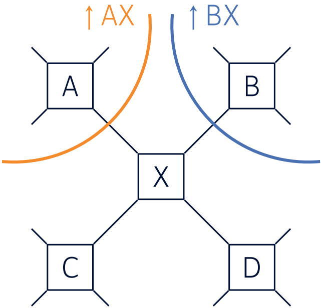

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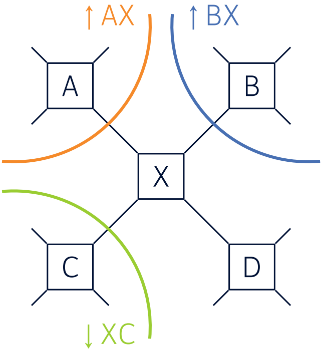

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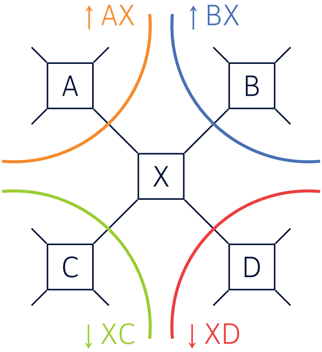

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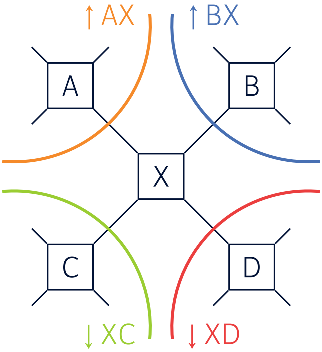

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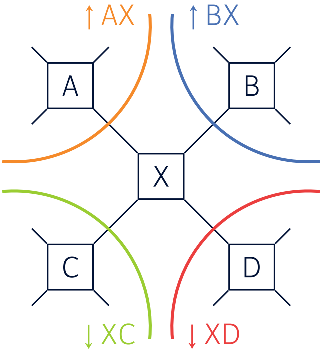

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  ↑ AX and ↑ BX are independent

Conclusion

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- **Bayesian networks**
  A memory-efficient way of storing a PMF by leveraging conditional independencies between variables

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- **Bayesian networks**
  A memory-efficient way of storing a PMF by leveraging conditional independencies between variables

- **Belief propagation**
  A time-efficient algorithm for computing the belief
  - Asynchronous, parallelizable
  - Exact in (poly)trees
  - In general, extended to the junction tree algorithm and to other (approximate) algorithms

