THE EXISTENCE OF FIXED POINTS FOR THE $\cdot/GI/1$ QUEUE

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The Annals of Probability (2003)

LINCS Network Theory Reading Group

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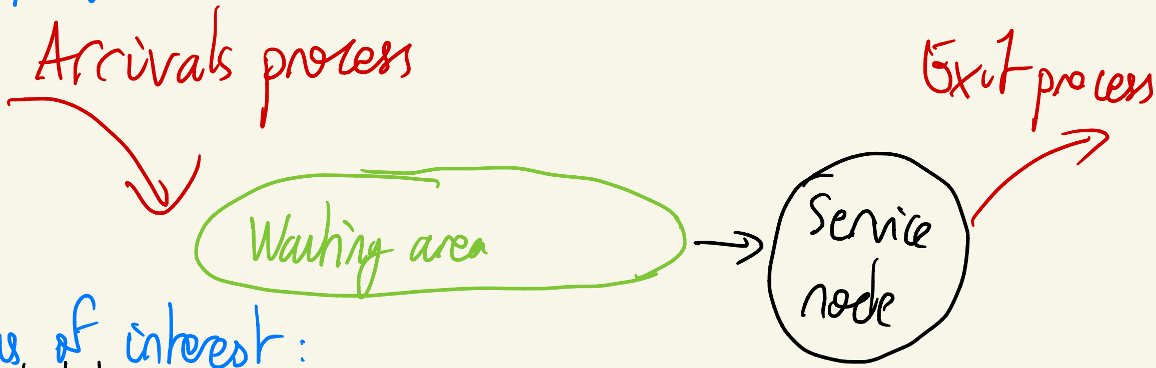
0. Queuing theory prerequisites

- Queuing model :



- Questions of interest :
  - → stability
  - → stationarity
- Kendall's notation

$$A / S / c \,(/ K / N / D)$$

- A: time between arrivals
- S: service time distribution
- c: number of service channels
- K: capacity of the queue (default: ∞)
- N: # of jobs to be served (default: ∞)
- D: queuing discipline (default: FIFO)

- Examples

→ M / M / 1

↳ Markovian → PPP (exponential inter-arrival times)

↳ Markovian → Exponential service time

↳ 1 → One service channel

→ M / GI / 1

↳ General independent distribution (i.i.d. service times)

Hereafter we are interested in a $\cdot$/GI/1 (/∞/FCFS) queue.

1. Motivation and main result

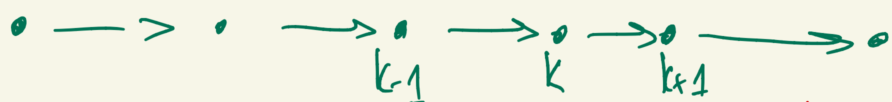
- **Goal:**

Existence of fixed points for $\cdot/GI/1$ queues, i.e. inter-arrival processes with the same distribution as the corresponding inter-departure process.

- **Interest:**

Limiting behavior of the distribution of departure processes from a tandem of queues.

Queues



$S(n,k)$ – service time of customer $n \in \mathbb{Z}$ in queue $k \in \mathbb{N}$

$A(n,k)$ – inter-arrival times between customers n and $n+1$.

- Assumptions :
  - $A^0 = (A(n,0), n \in \mathbb{Z})$ is ergodic and independent of $(S(n,k), n \in \mathbb{Z}, k \in \mathbb{N})$
  - $(S(n,k))$ are iid r.v.
  - $\mathbb{E}[S(0,0)] < \mathbb{E}[A(0,0)] < \infty$
  - $\mathbb{P}(S(0,0) \neq \mathbb{E}[S(0,0)]) > 0$

- Known result (Loynes) :

  For $k \geq 1$, each of the equilibrium departure processes $A^k = (A(n,k))_{n \in \mathbb{Z}}$ is ergodic of mean $\mathbb{E}[A(0,0)]$

· Questions :

→ Existence : does there exist a mean α ergodic inter-arrival process such that the interdeparture process has the same distribution?

If yes, we call it an ergodic fixed point of mean α

→ Uniqueness (if existence) of ergodic fixed point

→ Convergence : assume there is a unique ergodic fixed point of mean α. If A^0 is ergodic of mean α, does the law of A^k converge weakly to the ergodic fixed point as $k \to \infty$?

If yes, the fixed point is called an attractor

- Known results

- For exponential server queues:

Burke : Poisson process of rate $\frac{1}{\alpha}$ is a fixed point for exponential server queues with mean service $\beta < \alpha$.

· Anantharam : Uniqueness

· Mountford / Prabhakar : attractor

- For $\cdot/GI/1$ queues :

Chang : uniqueness

Prabhakar : uniqueness + convergence assuming finite mean (and existence)

- Main result of the paper

If the service time S has mean β and if $\int \mathbb{P}(S \geq v)^{\frac{1}{2}} dv < \infty$, then there is a set $\mathcal{S}$ closed in (β, ∞) with $\inf\{v \in \mathcal{S}\} = \beta$, $\sup\{v \in \mathcal{S}\} = \infty$ s.t.

(a) for $\alpha \in \mathcal{S}$, there exists a mean α ergodic fixed point for the queue

(b) For $\alpha \notin \mathcal{S}$, consider the stationary (but not ergodic) process F of mean α obtained as the convex combination of the ergodic fixed points of means $\underline{\alpha}$ and $\bar{\alpha}$, where $\underline{\alpha} = \sup\{v \in \mathcal{S}, v \leq \alpha\}$ and $\bar{\alpha} = \inf\{v \in \mathcal{S}, v \geq \alpha\}$. (Since $\mathcal{S}$ is closed, $\underline{\alpha}$ and $\bar{\alpha} \in \mathcal{S}$ and F is a fixed point for the queue).

If the inter-arrival times of the input process have a mean α, then the Cesaro average of the laws of the first k inter-departure processes converges weakly to F as $k \to \infty$.

Conjecture: $\mathcal{S} = (\beta, \infty)$ and (b) doesn't matter

2. Formalism

- The $\cdot/\cdot/1$ queue

We define the mappings 1) $\Psi : \mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} \longrightarrow \mathbb{R}_+^{\mathbb{Z}} \cup \{(+\infty)^{\mathbb{Z}}\}$

$$(a, s) \longmapsto w = \Psi(a, s)$$

inter-arrivals services workloads

with

$$w(n) = \Psi(a,s)(n) = \left[\sup_{j \leq n-1} \sum_{i=j}^{n-1} (s(i) - a(i))\right]^+$$

Note that this implies (Lindley's equation)

$$w(n) = [w(n-1) + s(n-1) - a(n-1)]^+$$

2) $\Phi : \mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$

$$(a,s) \longmapsto d = \Phi(a,s)$$

$\leftarrow$ inter-departures

with $d(n) = \Phi(a,s)(n) = [a(n) - s(n) - \Psi(a,s)(n)]^+ + s(n+1)$

Let $L : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$ denote the translation shift : $Lv(n) = v(n+1)$.

Then $d = [a - s - \Psi(a,s)]^+ + Ls$

When $w \in \mathbb{R}_+^{\mathbb{Z}}$, $d(n) = a(n) + w(n+1) - w(n) + s(n+1) - s(n)$

<u>Prop</u>: $\forall a,b \in \mathbb{R}_+^{\mathbb{Z}}$, $\forall s \in \mathbb{R}_+^{\mathbb{Z}}$, $a \leq b \Rightarrow \Psi(a,s) \geq \Psi(b,s)$

$$\Phi(a,s) \leq \Phi(b,s)$$

"Increasing inter-arrival times increases inter-departure times and decreases workloads"

· The stationary queue

Consider a measurable and P-stationary shift $\theta: \Omega \to \Omega$. Consider the processes $A: \Omega \to \mathbb{R}_+^{\mathbb{Z}}$ and $S: \Omega \to \mathbb{R}_+^{\mathbb{Z}}$ that are assumed compatible with θ and have a finite and positive mean.

(*) Set $W = \Psi(A, S)$ and $D = \Phi(A, S)$.

This model is a <u>stationary queue</u>. If θ is ergodic, the model is an ergodic queue.

When S is iid and non constant, it is an i.i.d. queue.

(Loynes):

· When θ is ergodic, on the event $E[S(0)] < E[A(0)]$, $W \in \mathbb{R}_+^{\mathbb{Z}}$ and $E[D(0)] = E[A(0)]$ (stable)

on the event $E[S(0)] > E[A(0)]$, $W = \infty^{\mathbb{Z}}$ and $D(n) = S(n)$ $\forall n$ (unstable)

on the event $E[S(0)] = E[A(0)]$, anything can happen (critical)

Let σ be the law of S. Define $\Phi_\sigma : \mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}}) \to \mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$,

$$\mu \to \Phi_\sigma(\mu)$$

where $\Phi_\sigma(\mu)$ is the law of $\Phi(A,S)$ where $A\sim\mu$, $S\sim\sigma$ and $A \perp\!\!\!\perp S$.

The map Φ_σ is called the <u>queueing map</u>.

A distribution μ such that $\Phi_\sigma(\mu)=\mu$ is called a fixed point for the queue.

(Loynes): $\forall \alpha > \beta$, $\Phi_\sigma : \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}}) \to \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$, where β is the mean service time

$\forall \alpha \leq \beta$, $\Phi_\sigma : \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}}) \to \{\sigma\}$

- Stable i.i.d. queues in tandem

Let $\{S(n,k),\ n\in\mathbb{Z},\ k\in\mathbb{N}\}$ be iid $\mathbb{R}^+$-valued r.v.s with $\mathbb{E}[S(0,0)]=\beta\in\mathbb{R}_+^*$.

Assume $\mathbb{P}(S(0,0)=\beta)<1$.

For $k\in\mathbb{N}$, define $S^k:\Omega\to\mathbb{R}_+^{\mathbb{Z}}$ by $S^k=(S(n,k))_{n\in\mathbb{Z}}$. Let σ be the distribution of S^k. Consider $A^0=(A(n,0))_{n\in\mathbb{Z}}:\Omega\to\mathbb{R}_+^{\mathbb{Z}}$ and assume A^0 is stationary, independent of S^k for all k and satisfies $\mathbb{E}[A(0,0)]=\alpha\in\mathbb{R}_+^*$.

Let θ be a $\mathbb{P}$-stationary shift s.t. A^0 and S^k $\forall k$ are compatible with θ.

Let $\mathcal{I}$ be the corresponding invariant σ-algebra. We assume $\beta<\mathbb{E}[A(0,0)\mid\mathcal{I}]$ a.s.

STABILITY

Define for all $k \in \mathbb{N}$

$$W^k = (W(n,k))_{n\in\mathbb{Z}} = \Psi(A^k, S^k)$$

$$A^{k+1} = (A(n,k+1))_{n\in\mathbb{Z}} = \Phi(A^k, S^k)$$

$$\left.\begin{array}{ll} A^k & \text{inter-arrival} \\ S^k & \text{service} \\ W^k & \text{workload} \end{array}\right\} \text{processes at queue } k.$$

A^{k+1} : inter-departure process at queue k and inter-arrival process at queue $k+1$.

The sequence $(A^k)_k$ is a Markov chain. μ is a stationary distribution of (A^k) iff μ is a fixed point for the queue. Does (A^k) admit nontrivial stationary distributions?

3. The results

· Uniqueness of fixed point (from existing literature)

Let $\mathcal{L}_s(\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}})$ be the set of random processes $((X(n), Y(n))_{n\in\mathbb{Z}}$ stationary in n.

Consider μ, ν in $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$ and let $\mathcal{D}(\mu,\nu) = \{(X,Y) \in \mathcal{L}_s(\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}}) \mid X \sim \mu, Y \sim \nu\}$

We consider the distance $\bar{\rho}(\mu,\nu) = \inf_{(X,Y)\in\mathcal{D}(\mu,\nu)} \mathbb{E}[|X(0) - Y(0)|]$.

<u>Thm</u>: Consider a stationary queue as defined in (*) with service process S and two inter-arrival processes A and B (with possibly different means). Assume $A \perp\!\!\!\perp S$ and $B \perp\!\!\!\perp S$.

Then $\bar{\rho}(\Phi(A,S), \Phi(B,S)) \leq \bar{\rho}(A,B)$.

In the iid service case: $<$

Thm: Consider a stable i.i.d. tandem model with interarrival processes A^0 and B^0 with different laws but such that $\mathbb{E}[A(0,0)|\mathcal{I}] = \mathbb{E}[B(0,0)|\mathcal{I}]$ a.s.

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Prabakhar

Recall $A^{n+1} = \Phi(A^n, S^n)$, $B^{n+1} = \Phi(B^n, S^n)$. Then, there exists $k \in \mathbb{N}^*$ s.t.

$$\bar{\rho}(A^k, B^k) < \bar{\rho}(A^0, B^0).$$

If $B^1 = \Phi(B^0, S^0) \sim B^0$, then $\lim_{n\to\infty} \bar{\rho}(A^n, B^0) = 0$ and $A^n \Rightarrow B^0$.

Let $\mathcal{M}_s^{p:\alpha}(\mathbb{R}_+^2) = \left\{ \mu \in \mathcal{M}_s^{\alpha}(\mathbb{R}_+^2) \;\middle|\; X \sim \mu \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X(i) = \alpha \text{ a.s.} \right\}$

Cor: 1) Consider an i.i.d. queue. The corresponding queueing map Φ_σ has at most one fixed point in $\mathcal{M}_s^{p:\alpha}(\mathbb{R}_+^2)$ for $\alpha > \mathbb{E}[S(0)]$

2) Consider an i.i.d queue and $\alpha > \mathbb{E}[S(0)]$. If $\zeta \in \mathcal{M}_s^{p:\alpha}(\mathbb{R}_+^2)$ is a fixed point, then it is necessarily ergodic (i.e. $\zeta \in \mathcal{M}_e^{\alpha}(\mathbb{R}_+^2)$).

· Existence of fixed point

Thm (Main result): Consider a queue with an i.i.d. service process S satisfying
$$\mathbb{E}[S(0)]\in\mathbb{R}_+^*,\quad \mathbb{P}(S(0)=\mathbb{E}[S(0)])<1 \text{ and } \int \mathbb{P}(S(0)\geq u)^{\frac{1}{2}}\,du<\infty.$$
Then there exists an ergodic inter-arrival process A with $A \perp\!\!\!\perp S$ and $\mathbb{E}[S(0)]<\mathbb{E}[A(0)]<\infty$ such that the corresponding inter-departure process D has the same distribution as A.

· Idea of proof

1) Consider Cesaro averages of the laws of A^k.

→ Consider the quadruple (A^k, S^k, W^k, A^{k+1}) denote its law by ν_k.
For $n \in \mathbb{N}^*$, define $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k$

→ The sequence $(\mu_n)_n$ is tight.

→ Let μ be a subsequential limit of $(\mu_n)_n$. Consider $(\hat{A}, \hat{S}, \hat{W}, \tilde{D}) \sim \mu$.

→ By the continuous mapping theorem and properties of Cesaro averages, we show
$\hat{A} \sim \tilde{D}$

2) Show that $\tilde{D} = \Phi(\hat{A}, \hat{S})$ using Birkhoff's ergodic theorem

3) Find conditions ensuring this fixed point is not the trivial fixed point 0.
↳ Uses the i.i.d. assumption

- Values of the means for which a fixed point exists

Consider a tandem of stable i.i.d. queues, let Φ_T be the corresponding queueing operator

Assume $\int_{\mathbb{R}_+} \mathbb{P}(S(0,0) \geq v)^{\frac{1}{2}} dv < \infty$.

Define $\mathcal{S} = \{\alpha \in (\beta, +\infty) \mid \exists \mu \in \mathcal{M}_e^\alpha(\mathbb{R}_+^2), \Phi_T(\mu) = \mu\}$

- From previous thm: $\mathcal{S} \neq \emptyset$
- Conjecture: $\mathcal{S} = (\beta, +\infty)$
- Actual result: $\mathcal{S}$ is unbounded and closed in (β, ∞).

For $\alpha \in \mathcal{S}$, denote by $\left|\begin{array}{l} \zeta_\alpha \text{ the unique ergodic fixed point of mean } \alpha \\ A_\alpha \text{ an inter-arrival process distributed as } \zeta_\alpha \end{array}\right.$

<u>Prop</u>: Consider an ergodic inter-arrival process A^0 of mean α.

1) If $\alpha \in \mathcal{S}$: $\bar{\rho}(A^k, A_\alpha) \xrightarrow[k\to\infty]{} 0$ and $A^k \Rightarrow A_\alpha$

2) If $\alpha \notin \mathcal{S}$, then $\frac{1}{k} \sum_{i=0}^{k-1} \mathcal{L}(A^i) \Rightarrow p\mathcal{L}(A_{\underline{\alpha}}) + (1-p)\mathcal{L}(A_{\bar{\alpha}})$ where

$$\underline{\alpha} = \sup\{u \in \mathcal{S}, u \leq \alpha\}, \quad \bar{\alpha} = \inf\{u \in \mathcal{S}, u \geq \alpha\}$$

and $$p = \frac{\bar{\alpha} - \alpha}{\bar{\alpha} - \underline{\alpha}}$$

THANK YOU FOR

LISTENING!