

Multi-winner voting rules

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Voting rules in general

Voters, candidates (= options to choose from).

Input = ballots, typically:

- Rankings (complete or not, with ties or not),
- Grades,
- Approvals.

Output:

- One candidate (single-winner rules),
- Several candidates (multi-winner rules):
 - Committee of fixed size k (this talk),
 - Committee of variable size,
- Ranking over the candidates (social welfare functions).

Multi-winner voting rules: old and new problems

Problems that already exist in single-winner rules:

- Condorcet paradox,
- Arrow theorem,
- Gibbard-Satterthwaite theorem.

These problems still exist for multi-winner rules.

But we also have new problems:

- What objective do we pursue?
- Computational complexity of computing the winners.

Candidates: $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2$.

Voters	73	23	2	2
Approvals	A_1,A_2,A_3,A_4	B_1, B_2	C ₁	D ₁

Candidates: $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2$.

Voters	73	23	2	2
Approvals	A_1,A_2,A_3,A_4	B_1, B_2	C ₁	D ₁

Objective	Example of scenario	Winners
Excellence	Recruit $k = 4$ taxi drivers	$\{A_1, A_2, A_3, A_4\}$

Candidates: $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2$.

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Objective	Example of scenario	Winners
Excellence	Recruit $k = 4$ taxi drivers	$\{A_1, A_2, A_3, A_4\}$
Proportionality	Elect a parliament of $k = 4$ members	$\{A_1, A_2, A_3, B_1\}$

Candidates: $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2$.

Voters	73	23	2	2
Approvals	A_1,A_2,A_3,A_4	B_1, B_2	C ₁	D ₁

Objective	Example of scenario	Winners
Excellence	Recruit $k = 4$ taxi drivers	$\{A_1, A_2, A_3, A_4\}$
Proportionality	Elect a parliament of $k = 4$ members	$\{A_1, A_2, A_3, B_1\}$
Diversity	Choose locations for $k = 4$ defibrillators	$\{A_1, B_1, C_1, D_1\}$

Plan

Zoology of rules Best-k rules Committee scoring rules Other rules

Discussion

A word on computational complexity Which rule for which objective?

Conclusion

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Our running example

Voters	27	12	5	22	21	13
	А	С	С	D	E	E
	В	D	E	С	А	В
Rankings	С	Е	D	В	В	С
_	D	В	А	Е	D	A
	Е	А	В	А	С	D
Approvals	A,B,C,D	С	A,D,C,E	B,C,D,E	A,B,D,E	E

We want to elect a committee of size k = 2.

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Best-k Rules

Recipe

Take a single-winner voting rule that produces scores (or a ranking over the candidates).

Output the k candidates with the best scores.

Single Non-Transferable Voting (SNTV)

Principle: best k candidates by Plurality.

Voters	27	12	5	22	21	13
	Α	С	С	D	E	E
	В	D	E	С	А	В
Rankings	С	E	D	В	В	С
	D	В	А	Е	D	А
	Е	А	В	А	С	D

Example: $score(A) = 1 \times 27 = 27$.

Candidate	A	В	С	D	E
Score	27	0	17	22	34

Winning committee: $S = \{A, E\}$.

Bloc voting

Principle: best k candidates by k-approval (reminder: we consider k = 2).

Voters	27	12	5	22	21	13
	Α	С	С	D	E	E
	В	D	E	С	Α	В
Rankings	С	E	D	В	В	С
	D	В	А	E	D	А
	Е	А	В	А	С	D

Example: $score(A) = 1 \times 27 + 1 \times 21 = 48$.

Candidate	A	В	С	D	Е
Score	48	40	39	34	39

Winning committee: $S = \{A, B\}$.

best-k Borda

Principle: best k candidates by Borda rule.

Voters	27	12	5	22	21	13
	А	С	С	D	E	E
	В	D	Е	С	А	В
Rankings	С	E	D	В	В	С
	D	В	А	E	D	А
	Е	A	В	А	С	D

Example: score(A) = $4 \times 27 + 0 \times 12 + 1 \times 5 + 0 \times 22 + 3 \times 21 + 1 \times 13 = 189$.

Candidate	A	В	С	D	E
Score	189	218	214	182	197

Winning committee: $S = \{B, C\}$.

best-k Approval Voting

Principle: best k candidates by Approval Voting.

Example: score(A) = $1 \times 27 + 1 \times 5 + 1 \times 21 = 53$.

Candidate	А	В	С	D	Е
Score	53	70	66	75	61

Winning committee: $S = \{B, D\}$.

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Committee scoring rules

Recipe

Find a way to make each voter v assign a score to each possible committee S: $\label{eq:score} score_v(S).$

Output the committee with the best score.

N.B.: all the best-k rules seen before belong to this family. We have in this case:

$$score_v(S) = \sum_{c \in S} score_v(c).$$

Example on next slide...



best-k Borda, seen as a committee scoring rule

Reminders: • the winning committee was $S = \{B, C\}$,

• $score(S = \{B, C\}) = score(B) + score(C) = 432$.

Let us compute score($S = \{B, C\}$) another way:

Voters	27	12	5	22	21	13
	А	С	С	D	E	E
	В	D	E	С	А	В
Rankings	С	Е	D	В	В	С
_	D	В	А	Е	D	A
	Е	А	В	А	С	D
$score_v(S)$	5	5	4	5	2	5
$ v \cdot score_v(S)$	135	60	20	110	42	65

 \Rightarrow score(S = {B,C}) = 135 + 60 + 20 + 110 + 42 + 65 = 432.

Proportional Approval Voting (PAV)

Principle: $score_v(S) = 1 + 1/2 + ... + 1/i$, where i is the number of candidates in the committee S approved by voter v.

Winning committee: $S = \{C, D\}$ (believe me). For the example, let us compute score($S = \{C, D\}$):

Voters	27	12	5	22	21	13
Approvals	A, B, C , D	С	$A, \mathbf{D}, \mathbf{C}, E$	B, C , D , E	A,B,\mathbf{D},E	E
$score_v(S)$	1.5	1	1.5	1.5	1	0
$ v \cdot score_v(S)$	40.5	12	7.5	33	21	0

 \Rightarrow score(S = {C, D}) = 40.5 + 12 + 7.5 + 33 + 21 + 0 = 114.

Borda Chamberlin-Courant (a.k.a. just "Chamberlin-Courant") Principle: $score_v(S) = Borda_v(c)$, where c is the candidate that voter v likes best in the committee S.

Winning committee: $S = \{A, C\}$ (believe me). For the example, let us compute score($S = \{A, C\}$):

Voters	27	12	5	22	21	13
	Α	С	С	D	E	E
	В	D	E	С	Α	В
Rankings	С	E	D	В	В	С
	D	В	А	Е	D	A
	Е	А	В	А	С	D
$score_v(S)$	4	4	4	3	3	2
$ v \cdot score_v(S)$	108	48	20	66	63	26

 \Rightarrow score(S = {A, C}) = 108 + 48 + 20 + 66 + 63 + 26 = 331.

Approval Chamberlin-Courant (a.k.a. Approval-CC)

Principle: $score_v(S) = Approval_v(c)$,

where c is the candidate that voter v likes best in the committee S.

Winning committee: $S = \{C, E\}$ (believe me). For the example, let us compute score($S = \{C, E\}$):

Voters	27	12	5	22	21	13
Approvals	A, B, C , D	С	A, C, D, E	$B, \boldsymbol{C}, D, \boldsymbol{E}$	A, B, D, E	E
$score_v(S)$	1	1	1	1	1	1
$ v \cdot score_v(S)$	27	12	5	22	21	13

 \Rightarrow score(S = {C, E}) = 27 + 12 + 5 + 22 + 21 + 13 = 100.

Committee scoring rules: theory

 $\text{score}_v(c) = ?$

- Plurality (SNTV),
- k-approval (Bloc),
- Borda (k-Borda, Borda-CC),
- Approval (best-k Approval, PAV, Approval-CC).

 $score_v(S) = ?$

- $\sum_{c \in S} \text{score}_v(c)$ (best-k rules),
- $\sum_{i} \alpha_i \cdot \text{score}_v(c_i)$, where c_i is the i-th preferred candidate of v in S (PAV).
- $\max_{c \in S} \text{score}_v(c)$ (Chamberlin-Courant).

N.B.: all are particular cases of the second one, called order-weighted average.

 $\mathsf{score}(S) = \sum_v \mathsf{score}_v(S)$ (but we could choose otherwise).

Committee scoring rules: sum-up table

	$score_v(S) =$					
$score_v(c) =$	$sum_{c\in S}score_v(c)$	$\sum_{i} \alpha_{i} \cdot score_{v}(c_{i})$	$\max_{c \in S} score_v(c)$			
Plurality	SNTV					
k-approval	Bloc					
Borda	best-k Borda		Borda-CC			
Approval	best-k Approval	PAV	Approval-CC			

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Not all multi-winner voting rules are committee scoring rules!





Iterated single-winner rules

- Elect one candidate by the single-winner rule.
- Remove her from the ballots and iterate.

Example: with Plurality.

Voters	27	12	5	22	21	13
	Α	С	С	D	E	E
	В	D	E	С	А	В
Rankings	С	Е	D	В	В	С
	D	В	А	Е	D	А
	Е	А	В	А	С	D

 $\Rightarrow \mathsf{Winners} = \{ \ , \ \}.$

Iterated single-winner rules

- Elect one candidate by the single-winner rule.
- Remove her from the ballots and iterate.

Example: with Plurality.

Voters	27	12	5	22	21	13
	Α	С	С	D		
	В	D		С	Α	В
Rankings	С		D	В	В	С
	D	В	А		D	А
		А	В	А	С	D

 $\Rightarrow Winners = \{ \ , E\}.$

Iterated single-winner rules

- Elect one candidate by the single-winner rule.
- Remove her from the ballots and iterate.

Example: with Plurality.

Voters	27	12	5	22	21	13
	Α	С	С	D		
	В	D		С	Α	В
Rankings	С		D	В	В	С
	D	В	А		D	А
		А	В	А	С	D

 \Rightarrow Winners = {A, E}.

- $Quota_k = \frac{V}{k+1}$. Ex: $Quota_1 = 50$, $Quota_2 = 33.3$, $Quota_3 = 25...$
- Elect all candidates with more than Quota_k top-votes and remove Quota_k voters who vote for each of them (see below). Iterate.
- If no candidate has the quota, eliminate the candidate with least top-votes.

Voters	27	12	5	22	21	13
	Α	С	С	D	E	E
	В	D	E	С	А	В
Rankings	С	E	D	В	В	С
	D	В	А	E	D	А
	E	А	В	А	С	D

$$\Rightarrow$$
 Winners = { , }.

- $Quota_k = \frac{V}{k+1}$. Ex: $Quota_1 = 50$, $Quota_2 = 33.3$, $Quota_3 = 25...$
- Elect all candidates with more than Quota_k top-votes and remove Quota_k voters who vote for each of them (see below). Iterate.
- If no candidate has the quota, eliminate the candidate with least top-votes.

Voters	27	12	5	22	0.41	0.25
	Α	С	С	D		
	В	D		С	Α	В
Rankings	С		D	В	В	С
	D	В	А		D	А
		А	В	А	С	D

$$\Rightarrow$$
 Winners = { , E}.

- $Quota_k = \frac{V}{k+1}$. Ex: $Quota_1 = 50$, $Quota_2 = 33.3$, $Quota_3 = 25...$
- Elect all candidates with more than Quota_k top-votes and remove Quota_k voters who vote for each of them (see below). Iterate.
- If no candidate has the quota, eliminate the candidate with least top-votes.

Voters	27	12	5	22	0.41	0.25
	Α			D		
	В	D			Α	В
Rankings			D	В	В	
-	D	В	А		D	А
		А	В	А		D

$$\Rightarrow$$
 Winners = { , E}.

- $Quota_k = \frac{V}{k+1}$. Ex: $Quota_1 = 50$, $Quota_2 = 33.3$, $Quota_3 = 25...$
- Elect all candidates with more than Quota_k top-votes and remove Quota_k voters who vote for each of them (see below). Iterate.
- If no candidate has the quota, eliminate the candidate with least top-votes.

Voters	27	12	5	22	0.41	0.25
	Α			D		
	В	D			Α	В
Rankings			D	В	В	
-	D	В	А		D	А
		А	В	А		D

 \Rightarrow Winners = {D, E}.

Condorcet rules

Principle: if there exists S of size k such that any candidate in S beats any candidate out of S, then S must be selected.

Weighted majority matrix of our example:

	А	В	С	D	Е
А		53	48	61	27
В	47		61	61	49
С	52	39		57	66
D	39	39	43		61
Е	73	51	34	39	

Here there is no such set S, because $A >_{Maj} B >_{Maj} C >_{Maj} D >_{Maj} E >_{Maj} A$. The winning committee will depend on the particular Condorcet rule we use (beyond the scope of this talk).

Borda Monroe (a.k.a. just "Monroe")

Variant of Chamberlin-Courant ensuring that not too many voters are "represented" by the same candidate. Beyond the scope of this talk.

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A word on computational complexity Which rule for which objective?

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NOKIA Bell Labs

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A word on computational complexity

Not computable in polynomial time:

- PAV,
- Monroe (in general),
- Chamberlin-Courant (in general).

Sequential variant: start from $S = \emptyset$ and add candidates one by one greedily.

Reverse sequential variant: start from $S = \{all the candidates\}$ and remove candidates one by one greedily.

Other approaches: fixed-parameter tractability (FPT), heuristics.

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New running example

Voters	66	12	11	10	1
	A ₁	B ₁	B ₂	B ₃	C ₁
	A ₂	B ₂	B ₁	B ₂	C ₂
	A ₃	B ₃	B ₃	B ₁	C ₃
Rankings	B ₁	A ₁	A ₁	A ₁	A ₁
	:	:	:	:	÷
	C ₁	C ₁	C ₁	C ₁	B ₁
		:		•	
Approvals	All A _i	All B _i	All B _i	All B _i	All C _i

Assumption: we want to elect k = 3 candidates.

Excellence

Intuition: select the "best" candidates based on some criterion.

 \Rightarrow An **individual** notion about each elected candidate (rather than a notion about the elected committee as a whole).

Examples:

Criterion	Voting rule
Number of approvals	best-k Approval
Borda score	best-k Borda
Being preferred by a majority of voters	Condorcet rules

Excellence: k-best Approval

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

Winners = any three A_i's (depending on the tie-breaking rule).

Rationale: each A_i is "better" than any non-A candidate, because more approved.

Excellence: Condorcet Rules

Voters	66	12	11	10	1
	A ₁	B ₁	B ₂	B ₃	C ₁
	A ₂	B ₂	B ₁	B ₂	C ₂
	A ₃	B ₃	B ₃	B ₁	C ₃
Dankings		:		:	÷
Rankings	B ₁	A ₁	A ₁	A ₁	A ₁
	:			•	:
	C ₁	C ₁	C ₁	C ₁	B ₁
	•			•	
Approvals	All A _i	All B _i	All B _i	All B _i	All C _i

Winners = $\{A_1, A_2, A_3\}$.

Rationale: each of them is "better" than (= preferred by a majority to) any non-elected candidate.

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Excellence: Concluding Remark

- The two rules in previous slides give (approximately) the same outcome.
- But for some other rules that can be defended as promoting "excellence", the outcome could be different: for example, k-best Plurality would elect $\{A_1, B_1, B_2\}$.
- Excellence is **not** a formally defined notion.

Proportionality

Intuition: more numerous voters should be "represented" by more candidates.

If voters and candidates can be partitioned into several (political) parties, such that all voters of a party prefers all candidates of their party to all other candidates, **then** each party should have a number of seats proportional to the number of voters in her party (up to roundings).

 \Rightarrow Proportionality **is** a formally defined notion that says what should be the outcome in some particular profiles (but not all of them).

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

 $\begin{array}{l} \Delta_S(A_i)=66\\ \Delta_S(B_i)=33\\ \Delta_S(C_i)=1 \end{array}$

 $\mathsf{Winners} = \{ \ , \ , \ \}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{split} \Delta_S(A_i) &= 66 \\ \Delta_S(B_i) &= 33 \\ \Delta_S(C_i) &= 1 \end{split}$$

\Rightarrow Elect A₁ (for example).

 $\text{Winners}=\{A_1, \quad, \quad \}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{array}{l} \Delta_S(A_i)=66/2=33\\ \Delta_S(B_i)=33\\ \Delta_S(C_i)=1 \end{array}$$

Here is the trick that makes PAV proportional: Adding a **second** A_i or a **first** B_i gives as many points.

 $Winners = \{A_1, \quad, \quad \}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{split} &\Delta_S(A_i)=66/2=33\\ &\Delta_S(B_i)=33\\ &\Delta_S(C_i)=1 \end{split}$$

Here is the trick that makes PAV proportional: Adding a **second** A_i or a **first** B_i gives as many points.

 $\Rightarrow \text{Elect } A_2 \text{ (for example).}$ $\text{Winners} = \{A_1, A_2, \dots\}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{split} \Delta_S(A_i) &= 66/3 = 22 \\ \Delta_S(B_i) &= 33 \\ \Delta_S(C_i) &= 1 \end{split}$$

 $Winners = \{A_1, A_2, \quad \}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{array}{l} \Delta_S(A_i)=66/3=22\\ \Delta_S(B_i)=33\\ \Delta_S(C_i)=1 \end{array}$$

\Rightarrow Elect B_1 (for example).

 $Winners = \{A_1, A_2, B_1\}.$

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

$$\begin{split} \Delta_S(A_i) &= 66/3 = 22 \\ \Delta_S(B_i) &= 33 \\ \Delta_S(C_i) &= 1 \end{split}$$

 \Rightarrow Elect B_1 (for example).

 $Winners = \{A_1, A_2, B_1\}.$

For k = 6, we would have 4 A_i's and 2 B_i's because:

 $\Delta_{S}(\text{fourth } A_{i}) = 66/4 = \Delta_{S}(\text{second } B_{i}) = 33/2.$

Voters	66	12	11	10	1
Rankings	A1	B 1	B₂	B ₃	C1
	A2	B₂	B1	B ₂	C2
	A3	B₃	B3	E ₁	C3
	:	:	:	∶	:
	B1	A₁	A1	A ₁	A1
	:	:	:	∶	:
	C1	C₁	C1	C ₁	B1
	:	:	:	∶	:

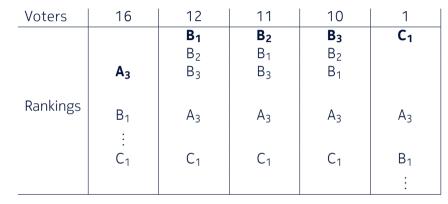
 $Winners = \{ \ , \ , \ \}.$

B1 B2 B3 C1 A2 B2 B1 B2 C2 A3 B3 B3 B1 C3 Rankings B1 A2 A2 A2	Voters	41	12	11	10	1
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	Rankings	A ₃ : B ₁ :	B ₂ B ₃ : A ₂ :	B ₁ B ₃ : A ₂ :	B ₂ B ₁ : A ₂ :	C ₂ C ₃ : A ₂ :

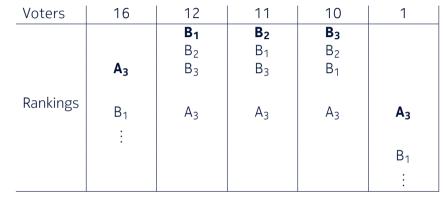
 $Winners = \{A_1, \quad, \quad \}.$

Voters	16	12	11	10	1
	A ₃	B 1 B ₂ B ₃	B 2 B1 B3	B 3 B2 B1	C ₁ C ₂ C ₃
Rankings	: B ₁ :	: A ₃ :	: A ₃ :	: A ₃ :	: A ₃ :
	С ₁	С ₁	С ₁	С ₁	В ₁
	:		:	:	÷

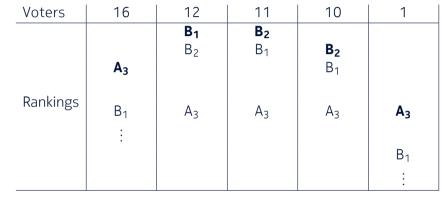
 $Winners = \{A_1, A_2, \}.$



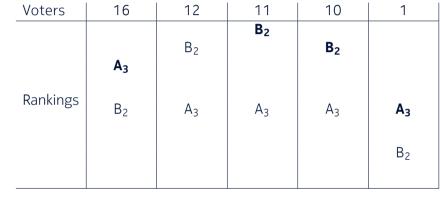
 $Winners = \{A_1, A_2, \quad \}.$



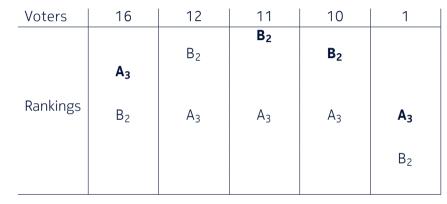
 $Winners = \{A_1, A_2, \dots\}.$



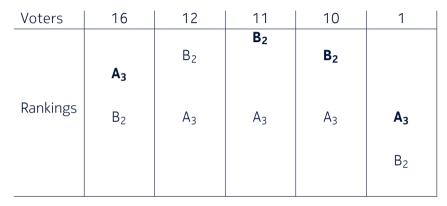
 $Winners = \{A_1, A_2, \dots\}.$



 $Winners = \{A_1, A_2, \dots\}.$



 $Winners = \{A_1, A_2, B_2\}.$



Winners = $\{A_1, A_2, B_2\}$. For k = 6, we would have $\{A_1, A_2, A_3, A_4, B_2, B_1\}$.



Intuition: as many voters as possible should be well "represented" by at least one candidate.

This is **not** a formally defined notion.



Diversity: Approval Chamberlin-Courant (Approval-CC)

Voters	66	33	1
Approvals	All A _i	All B _i	All C _i

 $Winners = \{any A_i, any B_i, any C_i\}.$

Two possible rationales:

- Once A-voters have one candidate A_i in the outcome, they are as happy as they can be.
- **Or** they could be more happy, but it is more important to represent as many voters as possible, including the only C-voter.

Diversity: Concluding Remark

Classic example to justify diversity: choosing movies for the catalogue of a short plane travel, because each passenger will watch only one movie. But...

Assume the following poll result for a sample of potential passengers:

Voters	54.4%	27.2%	18.1%	0.1%	0.1%	0.1%
Approvals	Genre A	Genre B	Genre C	Genre D	Genre E	Genre F

For k = 6, do you really want:

- One movie of each genre?
- Or give at least two possible choices for the people who like genre A?

 \Rightarrow Diversity is a very extreme point of view, giving a big power to arbitrary small minorities.

Summary: Which rule for which objective?

Arguably:

- **Excellence** (select "good" candidates):
 - Best-k rules, iterated single-winner rules, Condorcet rules.
- **Proportionality** (more voters should be represented by more candidates): PAV, STV, Monroe.
- **Diversity** (as many voters as possible should be represented): Borda-CC, Approval-CC.

In fact, since excellence and diversity are not formally defined, there are no clear frontiers between these three objectives...

Plan

Zoology of rules Best-k rules Committee scoring

Other rules

Discussion

A word on computational complexity Which rule for which objective?

Conclusion

Take-aways

- Multi-winner rules differ on their objective: **excellence**, **proportionality** or **diversity**.
- A large class of rules is given by the **committee scoring rules**.
- Some interesting rules are **computationally hard to compute**.

Bibliography: P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Multiwinner voting: A new challenge for social choice theory. In U. Endriss, editor, Trends in Computational Social Choice. Al Access, 2017.

Thanks For Your Attention!



