Fluid limits : a useful tool to assert stability conditions in queuing networks

Pierre Popineau

INRIA, ERC Nemo

November, 4, 2020

Outline of the presentation

1. An example: a M/G/1 interference network
2. Definitions on Markov chains
3. Preliminiary results
4. Fluid-scaling and fluid limits
   - Positive Harris recurrence
   - Application
5. A reciprocal: weak instability
   - Application to an example

A M/G/1 Queue

Let $\mathscr{D}$ be a bounded subset of $\mathbb{Z}^2$, with $m = |\mathscr{D}|$, and number the points of $\mathscr{D}$ from 0 to $m-1$.

A M/G/1 Queue

Let $\mathscr{D}$ be a bounded subset of $\mathbb{Z}^2$, with $m = |\mathscr{D}|$, and number the points of $\mathscr{D}$ from 0 to $m-1$.
Each point of $\mathscr{D}$ hosts a queue. Let $X_i(t)$ be the population of the i^{th} queue at time t. Let $(a_i)_{0 \le i \le m-1}$ a non-trivial sequence of $\mathbb{N}^m$.

A M/G/1 Queue

Let $\mathscr{D}$ be a bounded subset of $\mathbb{Z}^2$, with $m = |\mathscr{D}|$, and number the points of $\mathscr{D}$ from 0 to $m-1$.
Each point of $\mathscr{D}$ hosts a queue. Let $X_i(t)$ be the population of the i^{th} queue at time t. Let $(a_i)_{0\leq i\leq m-1}$ a non-trivial sequence of $\mathbb{N}^m$.

- Arrival process: let us suppose that arrival happen according to a Poisson point process with parameter $\lambda > 0$.

A M/G/1 Queue

Let $\mathscr{D}$ be a bounded subset of $\mathbb{Z}^2$, with $m = |\mathscr{D}|$, and number the points of $\mathscr{D}$ from 0 to $m-1$.
Each point of $\mathscr{D}$ hosts a queue. Let $X_i(t)$ be the population of the i^{th} queue at time t. Let $(a_i)_{0\le i\le m-1}$ a non-trivial sequence of $\mathbb{N}^m$.

- Arrival process: let us suppose that arrival happen according to a Poisson point process with parameter $\lambda > 0$.
- Departure process: each user have a file to transmit to the network, distributed exponentially with mean $L > 0$. Using a linearised Shannon-Hartley's formula, we get the instantaneous departure rate of queue j at time t as:

$$R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$$

A M/G/1 Queue

Let $\mathscr{D}$ be a bounded subset of $\mathbb{Z}^2$, with $m = |\mathscr{D}|$, and number the points of $\mathscr{D}$ from 0 to $m-1$.
Each point of $\mathscr{D}$ hosts a queue. Let $X_i(t)$ be the population of the i^{th} queue at time t. Let $(a_i)_{0 \le i \le m-1}$ a non-trivial sequence of $\mathbb{N}^m$.

- Arrival process: let us suppose that arrival happen according to a Poisson point process with parameter $\lambda > 0$.
- Departure process: each user have a file to transmit to the network, distributed exponentially with mean $L > 0$. Using a linearised Shannon-Hartley's formula, we get the instantaneous departure rate of queue j at time t as:

$$R_j(t) = \frac{1}{L} \frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$$

Goal : study the stability of underlying Markov chain (depending on λ)

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.
- State i is *transient* iff $\mathbb{P}[T_i < \infty | X_0 = i] = \sum_{n=1}^{\infty} f_{ii}^n < 1$.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.
- State i is *transient* iff $\mathbb{P}[T_i < \infty | X_0 = i] = \sum_{n=1}^{\infty} f_{ii}^n < 1$.
- State i is *positive recurrent* iff $\mathbb{E}[T_i] = \sum_{n=1}^{\infty} n f_{ii}^n < \infty$.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.
- State i is *transient* iff $\mathbb{P}[T_i < \infty | X_0 = i] = \sum_{n=1}^{\infty} f_{ii}^n < 1$.
- State i is *positive recurrent* iff $\mathbb{E}[T_i] = \sum_{n=1}^{\infty} n f_{ii}^n < \infty$.

X is said to be *transient* if all its states are transient, *recurrent* if none of it states are transients.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.
- State i is *transient* iff $\mathbb{P}[T_i < \infty | X_0 = i] = \sum_{n=1}^{\infty} f_{ii}^n < 1$.
- State i is *positive recurrent* iff $\mathbb{E}[T_i] = \sum_{n=1}^{\infty} n f_{ii}^n < \infty$.

X is said to be *transient* if all its states are transient, *recurrent* if none of it states are transients.

X is *ergodic* if it is aperiodic, and positive recurrent.

Definitions and Markov chains

Let X be a Markov chain with state space $\mathscr{S}$. We define:

- The *hitting time* of state $i \in \mathscr{S}$ is $T_i = \inf\{n \geq 1, X_n = i\}$.
- The *probability of return* to state i in n steps is $f_{ii}^n = \mathbb{P}[T_i = n | X_0 = i]$.
- State i is *transient* iff $\mathbb{P}[T_i < \infty | X_0 = i] = \sum_{n=1}^{\infty} f_{ii}^n < 1$.
- State i is *positive recurrent* iff $\mathbb{E}[T_i] = \sum_{n=1}^{\infty} n f_{ii}^n < \infty$.

X is said to be *transient* if all its states are transient, *recurrent* if none of it states are transients.

X is *ergodic* if it is aperiodic, and positive recurrent.
X is *stable* if it is ergodic with a unique stationary distribution.

Towards a stability criterion

Study the stability: drift arguments. Study the expected value of a jump in the state space.

Towards a stability criterion

Study the stability: drift arguments. Study the expected value of a jump in the state space.

Theorem (Foster, 1953)

Let X be a ϕ-irreducible discrete-time Markov chain. X is positive recurrent if and only if there exists a finite set C, a Lyapunov function V and constants $\alpha, \beta > 0$ such that:

$$\Delta V(\zeta) \equiv \mathbb{E}[V(X_1) - V(X_0) | X_0 = \zeta] \leq \beta \mathbb{1}\{\zeta \in C\} - \alpha \mathbb{1}\{\zeta \notin C\}$$

Towards a stability criterion

Study the stability: drift arguments. Study the expected value of a jump in the state space.

Theorem (Foster, 1953)

Let X be a ϕ-irreducible discrete-time Markov chain. X is positive recurrent if and only if there exists a finite set C, a Lyapunov function V and constants $\alpha, \beta > 0$ such that:

$$\Delta V(\zeta) \equiv \mathbb{E}[V(X_1) - V(X_0) | X_0 = \zeta] \leq \beta \mathbb{1}\{\zeta \in C\} - \alpha \mathbb{1}\{\zeta \notin C\}$$

Limitation : restricted to bounded jumps on the state space, which can be impractical to use (non-linear Lyapunov function, queueing disciplines).

Towards a stability criterion

Study the stability: drift arguments. Study the expected value of a jump in the state space.

Theorem (Foster, 1953)

Let X be a ϕ-irreducible discrete-time Markov chain. X is positive recurrent if and only if there exists a finite set C, a Lyapunov function V and constants $\alpha, \beta > 0$ such that:

$$\Delta V(\zeta) \equiv \mathbb{E}[V(X_1) - V(X_0) | X_0 = \zeta] \leq \beta \mathbb{1}\{\zeta \in C\} - \alpha \mathbb{1}\{\zeta \notin C\}$$

Limitation : restricted to bounded jumps on the state space, which can be impractical to use (non-linear Lyapunov function, queueing disciplines).

→ Is there a way to get a simple method to obtain the stability of a Markov chain ?

Fluid-scaling

Introduced by Rybko and Stolyar (1992). Instead of studying X, study the fluid-scaled process x defined as:

$$x_n(t) = \frac{1}{n} X^n(nt), \quad X^n(0) = n$$

Fluid-scaling

Introduced by Rybko and Stolyar (1992). Instead of studying X, study the fluid-scaled process x defined as:

$$x_n(t) = \frac{1}{n} X^n(nt), \quad X^n(0) = n$$

Main result of the paper: let X be a monoclass queuing network with FIFO discipline. Then:

Theorem (Rybko, Stolyar, 1992)

If there exists $T > 0$ such that $\forall t \geq T, \lim_{n\to\infty} \mathbb{E}[\|x_n(t)\|] = 0$ then X is ergodic.

Fluid-scaling

Introduced by Rybko and Stolyar (1992). Instead of studying X, study the fluid-scaled process x defined as:

$$x_n(t) = \frac{1}{n} X^n(nt), \quad X^n(0) = n$$

Main result of the paper: let X be a monoclass queuing network with FIFO discipline. Then:

Theorem (Rybko, Stolyar, 1992)

If there exists $T > 0$ such that $\forall t \geq T, \lim_{n\to\infty} \mathbb{E}[\|x_n(t)\|] = 0$ then X is ergodic.

$\lim_{n\to\infty} x_n(t) = \bar{x}(t)$ is called a *fluid limit* for the Markov chain X.

Fluid-scaling

Introduced by Rybko and Stolyar (1992). Instead of studying X, study the fluid-scaled process x defined as:

$$x_n(t) = \frac{1}{n} X^n(nt), \quad X^n(0) = n$$

Main result of the paper: let X be a monoclass queuing network with FIFO discipline. Then:

Theorem (Rybko, Stolyar, 1992)

If there exists $T > 0$ such that $\forall t \geq T, \lim_{n\to\infty} \mathbb{E}[\|x_n(t)\|] = 0$ then X is ergodic.

$\lim_{n\to\infty} x_n(t) = \bar{x}(t)$ is called a *fluid limit* for the Markov chain X.

Fluid limits prove to be useful to assert stability or instability for queuing networks.

Positive Harris recurrence

We need the following definition

Positive Harris recurrence

We need the following definition

Definition (Harris recurrence)

$\tau_A = \inf\{t \geq 0, X(t) \in A\}$ is the hitting time of A. X is *Harris recurrent* iff there exists μ, σ-finite such that $\mu(A) > 0$ and $A \subset \mathscr{S}$ imply $P(\tau_A < \infty | X_0 = x) = 1$, $\forall x \in \mathscr{S}$. X is *positive Harris recurrent* (PHR) iff it is Harris recurrent and its stationary distribution can be normalised to a probability distribution

Positive Harris recurrence

We need the following definition

Definition (Harris recurrence)

$\tau_A = \inf\{t \geq 0, X(t) \in A\}$ is the hitting time of A. X is *Harris recurrent* iff there exists μ, σ-finite such that $\mu(A) > 0$ and $A \subset \mathscr{S}$ imply $P(\tau_A < \infty | X_0 = x) = 1$, $\forall x \in \mathscr{S}$. X is *positive Harris recurrent* (PHR) iff it is Harris recurrent and its stationary distribution can be normalised to a probability distribution

Theorem (Meyn, Tweedie, 1993)

If there exists $\delta > 0$ such that

$$\lim_{x \to \infty} \frac{1}{|x|} \mathbb{E}\left[X(|x|\delta) | X_0 = x\right] = 0,$$

then $\sup_{x \in B}\{\mathbb{E}_x[\tau_B(\delta)]\} < \infty$, with $\tau_b(\delta) = \inf\{t \geq \delta, X(t) \in B\}$, and $B = B_1(0, \kappa)$ for some $\kappa > 0$. Consequently, X is PHR.

Positive Harris recurrence

We need the following definition

Definition (Harris recurrence)

$\tau_A = \inf\{t \geq 0, X(t) \in A\}$ is the hitting time of A. X is *Harris recurrent* iff there exists μ, σ-finite such that $\mu(A) > 0$ and $A \subset \mathscr{S}$ imply $P(\tau_A < \infty | X_0 = x) = 1$, $\forall x \in \mathscr{S}$. X is *positive Harris recurrent* (PHR) iff it is Harris recurrent and its stationary distribution can be normalised to a probability distribution

Theorem (Meyn, Tweedie, 1993)

If there exists $\delta > 0$ such that

$$\lim_{x \to \infty} \frac{1}{|x|} \mathbb{E}\left[X(|x|\delta) | X_0 = x\right] = 0,$$

then $\sup_{x \in B} \{\mathbb{E}_x[\tau_B(\delta)]\} < \infty$, with $\tau_b(\delta) = \inf\{t \geq \delta, X(t) \in B\}$, and $B = B_1(0, \kappa)$ for some $\kappa > 0$. Consequently, X is PHR.

→ Condition on the fluid-scaled model for positive Harris recurrence of the chain. Need to find a way to prove this condition systematically.

Positive Harris recurrence

We need the following definition

Definition (Harris recurrence)

$\tau_A = \inf\{t \geq 0, X(t) \in A\}$ is the hitting time of A. X is *Harris recurrent* iff there exists μ, σ-finite such that $\mu(A) > 0$ and $A \subset \mathscr{S}$ imply $P(\tau_A < \infty | X_0 = x) = 1$, $\forall x \in \mathscr{S}$. X is *positive Harris recurrent* (PHR) iff it is Harris recurrent and its stationary distribution can be normalised to a probability distribution

Theorem (Meyn, Tweedie, 1993)

If there exists $\delta > 0$ such that

$$\lim_{x \to \infty} \frac{1}{|x|} \mathbb{E}\left[X(|x|\delta) | X_0 = x\right] = 0,$$

then $\sup_{x \in B}\{\mathbb{E}_x[\tau_B(\delta)]\} < \infty$, with $\tau_b(\delta) = \inf\{t \geq \delta, X(t) \in B\}$, and $B = B_1(0, \kappa)$ for some $\kappa > 0$. Consequently, X is PHR.

→ Condition on the fluid-scaled model for positive Harris recurrence of the chain. Need to find a way to prove this condition systematically.

Fluid model and fluid limits

Let $\hat{X}$ be a fluid limit for our Markov chain. We introduce:

Fluid model and fluid limits

Let $\hat{X}$ be a fluid limit for our Markov chain. We introduce:

Definition (Stability)

A fluid limit is said to be *stable* if there exists $\delta > 0$ such that for any fluid limit with $|\hat{X}(0)| = 1$, we have $\hat{X}(\cdot + \delta) = 0$.

Fluid model and fluid limits

Let $\hat{X}$ be a fluid limit for our Markov chain. We introduce:

Definition (Stability)

A fluid limit is said to be *stable* if there exists $\delta > 0$ such that for any fluid limit with $|\hat{X}(0)| = 1$, we have $\hat{X}(\cdot + \delta) = 0$.

Using this definition, we can link PHR and stability of the system:

Fluid model and fluid limits

Let $\hat{X}$ be a fluid limit for our Markov chain. We introduce:

Definition (Stability)

A fluid limit is said to be *stable* if there exists $\delta > 0$ such that for any fluid limit with $|\hat{X}(0)| = 1$, we have $\hat{X}(\cdot + \delta) = 0$.

Using this definition, we can link PHR and stability of the system:

Theorem (Dai, 1995, [1])

If the fluid limit model for a fixed queuing discipline is stable, then the Markov chain X describing the dynamics of the network is PHR.

Fluid model and fluid limits

Let $\hat{X}$ be a fluid limit for our Markov chain. We introduce:

Definition (Stability)

A fluid limit is said to be *stable* if there exists $\delta > 0$ such that for any fluid limit with $|\hat{X}(0)| = 1$, we have $\hat{X}(\cdot + \delta) = 0$.

Using this definition, we can link PHR and stability of the system:

Theorem (Dai, 1995, [1])

If the fluid limit model for a fixed queuing discipline is stable, then the Markov chain X describing the dynamics of the network is PHR.

This gives a systematic method to check stability for a queuing network.

Systematic way to verify stability

Systematic way to verify stability

- Choose a representation for the queuing network (queue length $\mathbf{X}(t) = (X_i(t))$)

Systematic way to verify stability

- Choose a representation for the queuing network (queue length $\mathbf{X}(t) = (X_i(t))$)
- Obtain the temporal equation for the system (from system modelisation)

Systematic way to verify stability

- Choose a representation for the queuing network (queue length $\mathbf{X}(t) = (X_i(t))$)
- Obtain the temporal equation for the system (from system modelisation)
- Obtain the limit equation (functional laws of large numbers, deviation properties, tightness)

Systematic way to verify stability

- Choose a representation for the queuing network (queue length $\mathbf{X}(t) = (X_i(t))$)
- Obtain the temporal equation for the system (from system modelisation)
- Obtain the limit equation (functional laws of large numbers, deviation properties, tightness)
- Prove convergence of the fluid-scaling to the limit

Systematic way to verify stability

- Choose a representation for the queuing network (queue length $\mathbf{X}(t) = (X_i(t))$)
- Obtain the temporal equation for the system (from system modelisation)
- Obtain the limit equation (functional laws of large numbers, deviation properties, tightness)
- Prove convergence of the fluid-scaling to the limit
- Prove that the fluid limit reaches 0 (Lyapunov stability)

A M/G/1 interference network

Reminder:

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L} \frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathcal{A}_i)_{1 \le i \le m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1 \le i \le m}$ be Poisson point processes of intensity 1.
We get:

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathscr{A}_i)_{1\leq i\leq m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1\leq i\leq m}$ be Poisson point processes of intensity 1. We get:

$$X_i(t) =$$

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathscr{A}_i)_{1\leq i\leq m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1\leq i\leq m}$ be Poisson point processes of intensity 1. We get:

$$X_i(t) = \underbrace{X_i(0)}_{\text{Initial condition}}$$

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathscr{A}_i)_{1\leq i\leq m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1\leq i\leq m}$ be Poisson point processes of intensity 1. We get:

$$X_i(t) = \underbrace{X_i(0)}_{\text{Initial condition}} + \underbrace{\mathscr{A}_i(t)}_{\text{Arrival process}}$$

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathscr{A}_i)_{1\le i\le m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1\le i\le m}$ be Poisson point processes of intensity 1. We get:

$$X_i(t) = \underbrace{X_i(0)}_{\text{Initial condition}} + \underbrace{\mathscr{A}_i(t)}_{\text{Arrival process}} \underbrace{- N_i^s\left(\int_0^t \frac{1}{L}\frac{a_j X_j(s)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(s)} ds\right)}_{\text{Departure process}}.$$

A M/G/1 interference network

Reminder:

- Arrival process: Poisson point process of intensity $\lambda > 0$.
- Departure process: stochastic process with intensity $R_j(t) = \frac{1}{L}\frac{a_j X_j(t)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(t)}$.

Get the temporal evolution of the queue lengths. Let $(\mathscr{A}_i)_{1\le i\le m}$ be a Poisson point process of intensity λ and $(N_i^s)_{1\le i\le m}$ be Poisson point processes of intensity 1. We get:

$$X_i(t) = \underbrace{X_i(0)}_{\text{Initial condition}} + \underbrace{\mathscr{A}_i(t)}_{\text{Arrival process}} \underbrace{- N_i^s\left(\int_0^t \frac{1}{L}\frac{a_j X_j(s)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(s)} ds\right)}_{\text{Departure process}}.$$

Next step : establish the fluid equations

A M/G/1 interference network

For the arrival process: use Poisson point processes properties to get:

A M/G/1 interference network

For the arrival process: use Poisson point processes properties to get:

$$\frac{1}{n}\mathscr{A}_i(nt) \to \lambda t$$

A M/G/1 interference network

For the arrival process: use Poisson point processes properties to get:

$$\frac{1}{n}\mathscr{A}_i(nt) \to \lambda t$$

For the departure process: use tightness properties to get:

$$\frac{1}{n}N_i^s\left(\int_0^t \frac{1}{L}\frac{a_j X_j(s)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(s)} ds\right) \to \frac{1}{L}\int_0^s \frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s)} ds.$$

A M/G/1 interference network

For the arrival process: use Poisson point processes properties to get:

$$\frac{1}{n}\mathscr{A}_i(nt) \to \lambda t$$

For the departure process: use tightness properties to get:

$$\frac{1}{n}N_i^s\left(\int_0^t \frac{1}{L}\frac{a_j X_j(s)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(s)}ds\right) \to \frac{1}{L}\int_0^s \frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s)}ds.$$

To prove convergence: use Skorokhod representation theorem and use C-tightness (tight with almost surely continuous limits), cf [2]

A M/G/1 interference network

For the arrival process: use Poisson point processes properties to get:

$$\frac{1}{n}\mathscr{A}_i(nt) \to \lambda t$$

For the departure process: use tightness properties to get:

$$\frac{1}{n}N_i^s\left(\int_0^t \frac{1}{L}\frac{a_j X_j(s)}{\mathcal{N}_0 + \sum_{i=0}^{m-1} a_i X_i(s)} ds\right) \to \frac{1}{L}\int_0^s \frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s)} ds.$$

To prove convergence: use Skorokhod representation theorem and use C-tightness (tight with almost surely continuous limits), cf [2]
Fluid equation:

$$\bar{x}_i(t) = \bar{x}_i(0) + \lambda t - \frac{1}{L}\int_0^s \frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s) ds}.$$

Reduce the problem to a deterministic system of integral equations

A M/G/1 interference network

Differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ \bar{x}_i(0) = x_i^0 \end{cases}$$

A M/G/1 interference network

Differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds} \\ \bar{x}_i(0) = x_i^0 \end{cases}.$$

Let $V(x) = \sum_{i=0}^{m} x_j$. We immediately get:

A M/G/1 interference network

Differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ \bar{x}_i(0) = x_i^0 \end{cases}$$

Let $V(x) = \sum_{i=0}^{m} x_j$. We immediately get:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\bar{x}) = \lambda m - \frac{1}{L}.$$

A M/G/1 interference network

Differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ \bar{x}_i(0) = x_i^0 \end{cases}$$

Let $V(x) = \sum_{i=0}^{m} x_j$. We immediately get:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\bar{x}) = \lambda m - \frac{1}{L}.$$

Lyapunov stability: $\frac{\mathrm{d}}{\mathrm{d}t}V(\bar{x}) < 0$ which gives us a condition on the arrival rate:

$$\lambda < \frac{1}{Lm}.$$

A M/G/1 interference network

Differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s) ds}. \\ \bar{x}_i(0) = x_i^0 \end{cases}$$

Let $V(x) = \sum_{i=0}^{m} x_j$. We immediately get:

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\bar{x}) = \lambda m - \frac{1}{L}.$$

Lyapunov stability: $\frac{\mathrm{d}}{\mathrm{d}t} V(\bar{x}) < 0$ which gives us a condition on the arrival rate:

$$\lambda < \frac{1}{Lm}.$$

If $\lambda < \frac{1}{Lm}$, then $\bar{x}_i(t) \to 0$ as t goes to infinity, which implies that the queuing system is stable, i.e. that $\mathbf{X}$ is PHR.

Reciprocal

The stability of the fluid limit model implies the stability of the system. Does the instability of the fluid model leads to instability of the system ?

Reciprocal

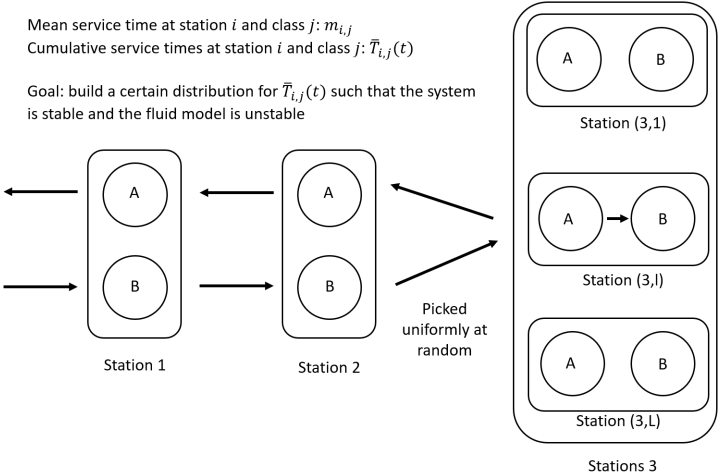
The stability of the fluid limit model implies the stability of the system. Does the instability of the fluid model leads to instability of the system ?

Answer: no. Bramson ([3]) built a network with two classes, an unstable fluid limit such that the queuing network is stable.

Reciprocal

The stability of the fluid limit model implies the stability of the system. Does the instability of the fluid model leads to instability of the system ?

Answer: no. Bramson ([3]) built a network with two classes, an unstable fluid limit such that the queuing network is stable.



Reciprocal

We need an *alternate* definition for instability:

Theorem (Dai, 1996, [4])

A fluid limit model is weakly unstable if there exists $\delta > 0$ *such that for each fluid solution* $\hat{Q}(\cdot)$ *starting from 0,* $\hat{Q}(\delta) \neq 0$. *If the fluid limit is weakly unstable, then we have with probability 1:*

$$\lim_{t \to \infty} |Q(t)| = +\infty$$

Reciprocal

We need an *alternate* definition for instability:

Theorem (Dai, 1996, [4])

A fluid limit model is weakly unstable if there exists $\delta > 0$ such that for each fluid solution $\hat{Q}(\cdot)$ starting from 0, $\hat{Q}(\delta) \neq 0$. If the fluid limit is weakly unstable, then we have with probability 1:

$$\lim_{t\to\infty} |Q(t)| = +\infty$$

Comes from the definition of *weak stability*: a model is weakly stable iff all fluid limits starting at 0 are trivial.

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

There exists $\delta(\omega) > 0$ such that $\forall \hat{Q}$ fluid limit, $\hat{Q}(\delta) > 0$.
Suppose that $\liminf_{r\to\infty} \left| \frac{Q(r\delta)}{r} \right| = 0$.

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

There exists $\delta(\omega) > 0$ such that $\forall \hat{Q}$ fluid limit, $\hat{Q}(\delta) > 0$.
Suppose that $\liminf_{r\to\infty} \left|\frac{Q(r\delta)}{r}\right| = 0$.

There exists a subsequence $\{r_n\}$ along which $\left|\frac{Q(r_n\delta)}{r_n}\right| \to 0$.

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

There exists $\delta(\omega) > 0$ such that $\forall \hat{Q}$ fluid limit, $\hat{Q}(\delta) > 0$.
Suppose that $\liminf_{r\to\infty} \left|\frac{Q(r\delta)}{r}\right| = 0$.

There exists a subsequence $\{r_n\}$ along which $\left|\frac{Q(r_n\delta)}{r_n}\right| \to 0$.
Moreover, $\{\frac{Q(r\cdot,\omega)}{r}, r \geq 1\}$ is precompact. There exists $\{r_{n_m}\}$ such that $\frac{Q(r_{n_m}\cdot)}{r_{n_m}}$ converges u.o.c. to $\hat{Q}$ fluid limit.

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

There exists $\delta(\omega) > 0$ such that $\forall \hat{Q}$ fluid limit, $\hat{Q}(\delta) > 0$.
Suppose that $\liminf_{r\to\infty} \left| \frac{Q(r\delta)}{r} \right| = 0$.

There exists a subsequence $\{r_n\}$ along which $\left| \frac{Q(r_n\delta)}{r_n} \right| \to 0$.
Moreover, $\{\frac{Q(r\cdot,\omega)}{r}, r \geq 1\}$ is precompact. There exists $\{r_{n_m}\}$ such that $\frac{Q(r_{nm}\cdot)}{r_{nm}}$ converges u.o.c. to $\hat{Q}$ fluid limit.

Thus $\left| \frac{Q(r_{nm}\delta)}{r_{nm}} \right| \to \left| \hat{Q}(\delta) \right| = 0$: **contradiction**

Outline of the proof

Assume that model is weakly unstable for a given sample path ω.

There exists $\delta(\omega) > 0$ such that $\forall \hat{Q}$ fluid limit, $\hat{Q}(\delta) > 0$.
Suppose that $\liminf_{r\to\infty} \left|\frac{Q(r\delta)}{r}\right| = 0$.

There exists a subsequence $\{r_n\}$ along which $\left|\frac{Q(r_n\delta)}{r_n}\right| \to 0$.
Moreover, $\{\frac{Q(r\cdot,\omega)}{r}, r \geq 1\}$ is precompact. There exists $\{r_{n_m}\}$ such that $\frac{Q(r_{nm}\cdot)}{r_{nm}}$ converges u.o.c. to $\hat{Q}$ fluid limit.

Thus $\left|\frac{Q(r_{nm}\delta)}{r_{nm}}\right| \to \left|\hat{Q}(\delta)\right| = 0$: **contradiction**

Hence, $\liminf_{r\to\infty} \left|\frac{Q(r\delta)}{r}\right| > 0$ implying that $\lim_{t\to\infty} |Q(t)| = +\infty$

A M/G/1 interference network

Fluid equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ \quad x_i(0) = x_i^0 \end{cases}$$

A M/G/1 interference network

Fluid equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s)ds}. \\ x_i(0) = x_i^0 \end{cases}$$

Assume that $\lambda > \frac{1}{Lm}$. We get:

A M/G/1 interference network

Fluid equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ x_i(0) = x_i^0 \end{cases}$$

Assume that $\lambda > \frac{1}{Lm}$. We get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{m-1}\bar{x}_i(t) = \lambda m - \frac{1}{L} > 0.$$

A M/G/1 interference network

Fluid equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \dfrac{1}{L}\dfrac{a_j \bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i \bar{x}_i(s) ds}. \\ x_i(0) = x_i^0 \end{cases}$$

Assume that $\lambda > \frac{1}{Lm}$. We get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{m-1}\bar{x}_i(t) = \lambda m - \frac{1}{L} > 0.$$

Thus, $\exists i$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) > 0$, i.e., $\exists \delta > 0$ such that $\bar{x}_i(\delta) > 0 \implies$ the model is weakly unstable.

A M/G/1 interference network

Fluid equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) = \lambda - \frac{1}{L}\frac{a_j\bar{x}_j(s)}{\sum_{i=0}^{m-1} a_i\bar{x}_i(s)ds}. \\ x_i(0) = x_i^0 \end{cases}$$

Assume that $\lambda > \frac{1}{Lm}$. We get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{m-1}\bar{x}_i(t) = \lambda m - \frac{1}{L} > 0.$$

Thus, $\exists i$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t) > 0$, i.e., $\exists\delta > 0$ such that $\bar{x}_i(\delta) > 0 \implies$ the model is weakly unstable.
We have proven:

$$\lambda < \frac{1}{Lm} \iff \mathbf{X} \text{ is stable.}$$

Conclusion

- Fluid limits are a way to systematically check a condition for PHR of a Markov chain

Conclusion

- Fluid limits are a way to systematically check a condition for PHR of a Markov chain
- Study the asymptotic regime of a queuing system over unbounded jumps

Conclusion

- Fluid limits are a way to systematically check a condition for PHR of a Markov chain
- Study the asymptotic regime of a queuing system over unbounded jumps
- Well-suited for open queuing networks

Conclusion

- Fluid limits are a way to systematically check a condition for PHR of a Markov chain
- Study the asymptotic regime of a queuing system over unbounded jumps
- Well-suited for open queuing networks
- Existence of a "reciprocal" to prove instability of the system

Conclusion

- Fluid limits are a way to systematically check a condition for PHR of a Markov chain
- Study the asymptotic regime of a queuing system over unbounded jumps
- Well-suited for open queuing networks
- Existence of a "reciprocal" to prove instability of the system

Thank you for your attention !

References I

J. G. Dai, "On positive harris recurrence of multiclass queuing networks: a unified approach via fluid limit methods," *The Annals of Applied Probability*, vol. 5, no. 1, pp. 49–77, 1995.

J. Reed and B. Zwart, "Limit theorems for markovian bandwidth sharing networks with rate constraints," *Operations Research*, 2014.

M. Bramson, "A stable queueing network with unstable fluid model," *The Annals of Applied Probability*, vol. 9, no. 3, pp. 818–853, 1999.

J. G. Dai, "A fluid limit model criterion for instability of multiclass queuing networks," *The Annals of Applied Probability*, vol. 6, no. 3, pp. 751–757, 1996.