Asymptotics for Euclidean minimal spanning trees on random points by David Aldous and J. Michael Steele

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Euclidean minimal spanning tree (MST)

Let $X = \{x_1, x_2, ..., x_n\}$ be points in $\mathbb{R}^d$ ($d \geq 2$). The minimal spanning tree (MST) t with vertex set X is a tree whose sum of the edge-lengths are minimum.

$$\sum_{e \in t} |e| = \min_G \sum_{e \in G} |e|$$

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- Degree of a vertex.
- Sum of the d-th powers of edge-lengths incident at a vertex. (d is the dimension)

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- For any relatively compact set B (i.e. closure(B) is compact), the number of points of $\mathcal{N}$ in B, $|\mathcal{N} \cap B|$ is a Poisson random variable with parameter $\Lambda(B)$.

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- For any finite number of pairwise disjoint relatively compact sets $B_1, B_2, ..., B_n$, the random variables $|\mathcal{N} \cap B_1|, ..., |\mathcal{N} \cap B_n|$ are independent.

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- Its Palm probability is equal to $\mathcal{N} \cup \delta_0 := \mathcal{N}^o$.

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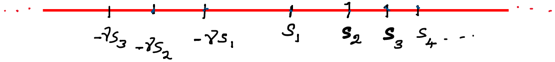
Examples

- The Poisson point process $\mathcal{N}$ is a.s. nice.
- Let $s_n = \sum_{i=1}^{n} 1/i$ and γ be an irrational number. $X_n = \{-\gamma s_n, ..., -\gamma s_1, s_1, ..., s_n\}$



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- $t_\infty(x) = \cup_{i=1}^{\infty} t_n(x)$.

Example

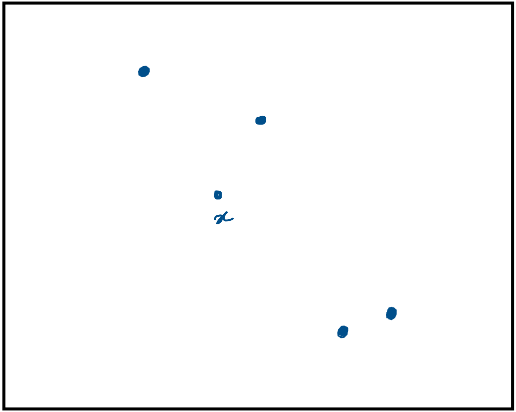

Figure: Construction of $t_n(x)$

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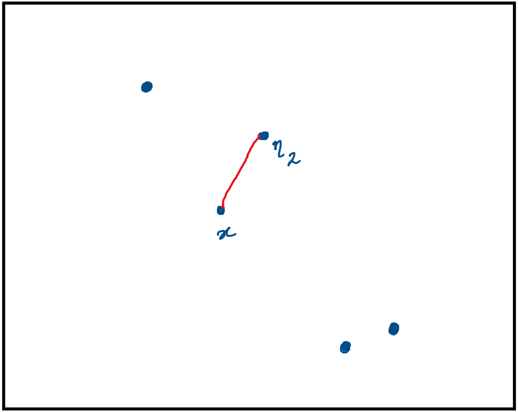

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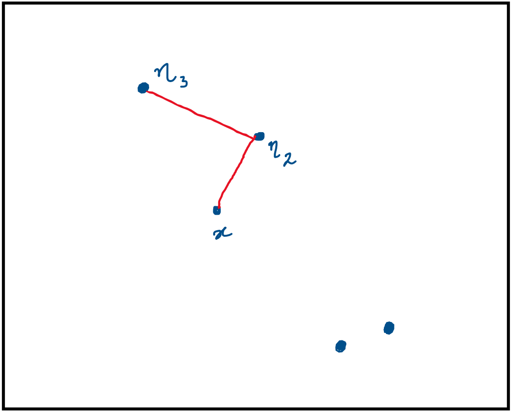

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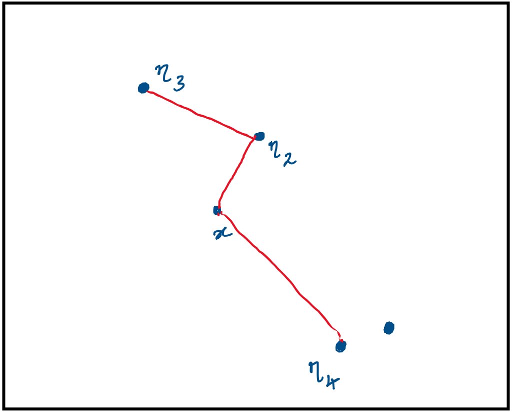

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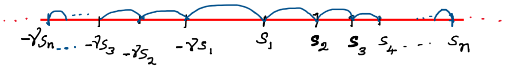

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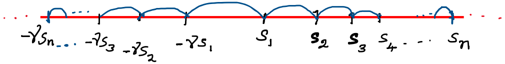
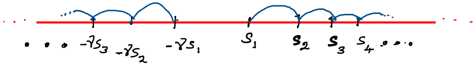


- For $X = \cup_{i \geq 1} X_n$, $g(X) =$



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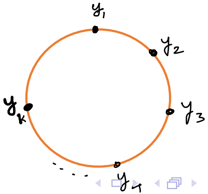
2. Let $x \in X$. Then by above observation $t_\infty(x) \subset g(X)$ proving that components are infinite.
3. Suppose there is a cycle $y_1, y_2, ...y_k, y_1$. Reorder them such that $|y_k - y_1|$ is the maximum among all distances. Then (y_1, y_k) is neither an edge $\in t_\infty(y_1)$ nor $\in t_\infty(y_k)$.



MSF of stationary Poisson point process, $g(\mathcal{N})$

Lemma

Let $\mathcal{N}^0 = \mathcal{N} \cup \delta_0$, $\mathcal{T}$ *be the connected component of* 0 *in* $g(\mathcal{N}^0)$, D *be the degree of* 0 *in* $\mathcal{T}$ *and* $L_1, L_2, ..., L_D$ *be the edge-lengths incident at* 0. *Then,*

1. $D \leq b_d$ *where* b_d *is a constant.*
2. $\mathbb{E}[D] = 2$.
3. $l_d = \sum_i \mathbb{E}[L_i^d] < \infty$.

Approximation of Poisson point process

Let $\mathcal{N}_n = \{\eta_1, \eta_2, ...\eta_n\}$ be i.i.d. points with uniform distribution on the unit cube $[0,1]^d$, $S_n = t_n(0, \mathcal{N}_n^*)$. Let $\mathcal{N}_n^* = \{n^{1/d}(\eta_i - \eta_1)\}$ be the scatter (scaled and shifted points) of $\mathcal{N}$. Then, we have the following proposition.

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Theorem

1. *$\mathcal{N}_n^* \to \mathcal{N}^0$ in distribution.*
2. *Let $\{e_i : i = 1, ...n-1\}$ be the edge-lengths of S_n. Then*

$$\sum_i^n |e_i|^d \to l_d \text{ in } L^2.$$

3. *Let $\Delta_{n,i}$ be the proportion of vertices of S_n with degree i, then for each i:*

$$\mathbb{E}[\Delta_{n,i}] \to \mathbb{P}[D = i].$$

Local convergence of finite sets

A sequence of nice sets X_n is said to converge locally to a nice set X is there exist a labelling of $X_n = \{x_{n1}, x_{n2}, ...\}$ and $X = \{x_1, x_2, ...\}$ such that:

1. $x_{ni} \to x_i$ for all i.
2. For any proper $C_L = [-L, L]^d$ (i.e., boundary of L does not intersect with X), $|X_n \cap C_L| \to |X \cap C_L|$.

Local convergence of graphs

Let $X_n = \{x_{n1}, x_{n2}, ...\}$ converge locally to $X = \{x_1, x_2, ...\}$, and h_n, h be graphs with vertex set X_n and X respectively, we say that h_n converge locally to h, if for any proper C_L, there exists $n_0 = n_0(L)$ such that for all $n \geq n_0$:

1. if (x_{ni}, x_{nj}) is an edge of h_n with $x_{ni} \in C_L$ then (x_i, x_j) is an edge of h.
2. if (x_i, x_j) is an edge of h with $x_i \in C_L$ then (x_{ni}, x_{nj}) is an edge of h_n.