Spatial birth-and-death wireless networks

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Introduction/motivation

- Densification of wireless networks (IoT, mobile phones, Bluetooth devices) -> need to study the traffic of wireless networks
- Already studied to some extent through queueing theory (Shannon)

A lot of connections depending on a lot of factors -> hard to compute

Does not completely model interference in the system

Instead: consider the network as the realization of a spatial dynamic random process

Outline of the presentation

- Point Processes
- Mathematical framework
  - Markov properties
- Stochastic geometry tools
- Proof of the main result
- Natural extensions
  - Random channel fading
  - Multivariate case

Point processes

- A stochastic process with values in $\mathbb{R}^d$ $(d \geq 1)$ is called a point process (PP)
- Can also be considered as random measures (random variables in a measure space)
- Used to model a variety of physical processes (trees in a forest, times of departure from a queueing system, etc.)

Point processes

- Poisson point process (PPP): the most convenient type of PP:
  - Number of points in B distributed according a Poisson law of parameter $\lambda|B|$
  - Mutual independence property: if $B_1, \dots, B_n$ are disjoint Borel sets, $\Phi_t(B_1), \dots, \Phi_t(B_n)$ are independent

- Intensity measure:

$$\Lambda(B) = \mathbb{E}[\Phi(B)]$$

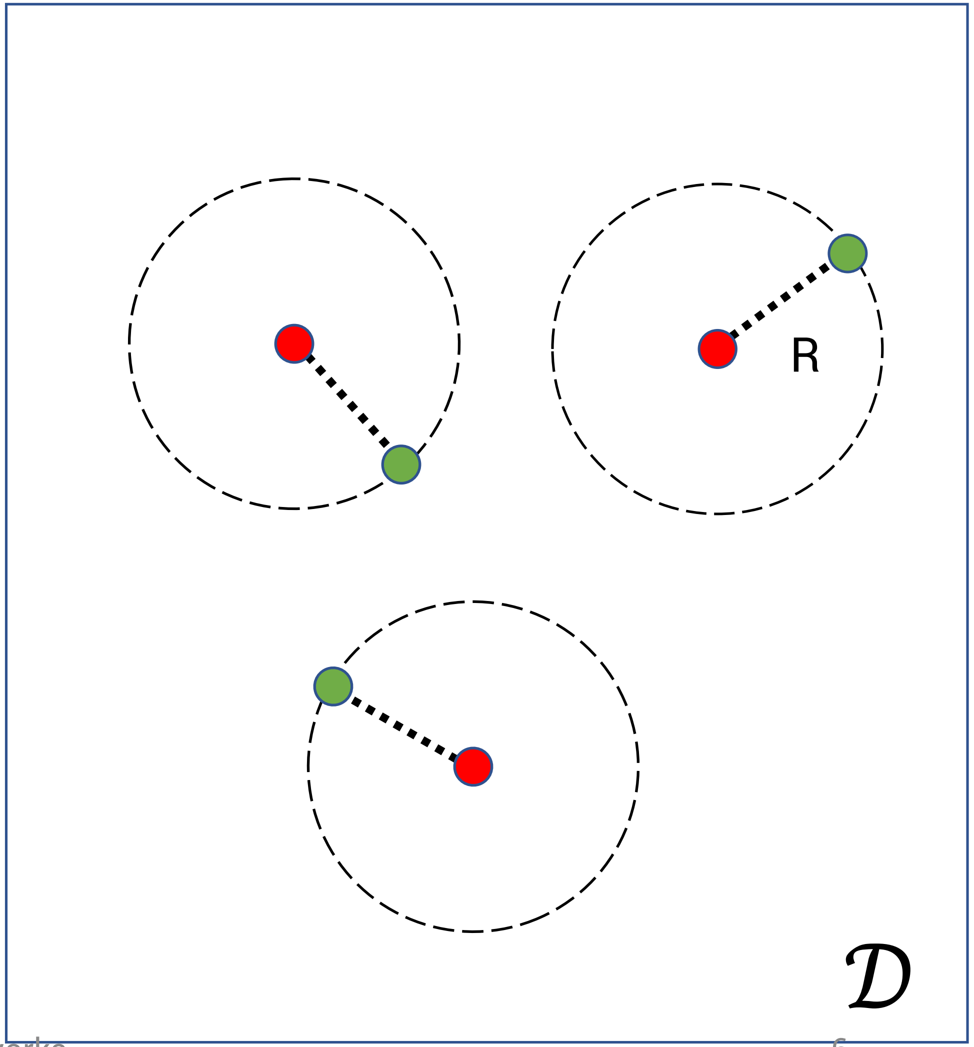
For a stationary point process:

$$\exists \alpha > 0, \qquad \Lambda = \alpha \nu_L$$

Mathematical framework

- Network: compact $\mathcal{D} \subset \mathbb{R}^2$
- Arrival of transmitters: Poisson rain of intensity $\lambda > 0$
- Each transmitter has a file of size distributed according $\mathcal{E}\left(\frac{1}{L}\right)$
- Receivers are located at distance $R > 0$ from transmitters
- Φ_t : point process of transmitters in the system at time t
- Point leaves the system when transmission is over

-> Poisson dipolar model



Mathematical framework

- Path-loss function ℓ: non-negative, bounded and integrable
- Interference experienced at point x in the system:

$$I(x, \Phi_t) = \sum_{y \in \Phi_t \setminus \{x\}} \ell(\|x - y\|)$$

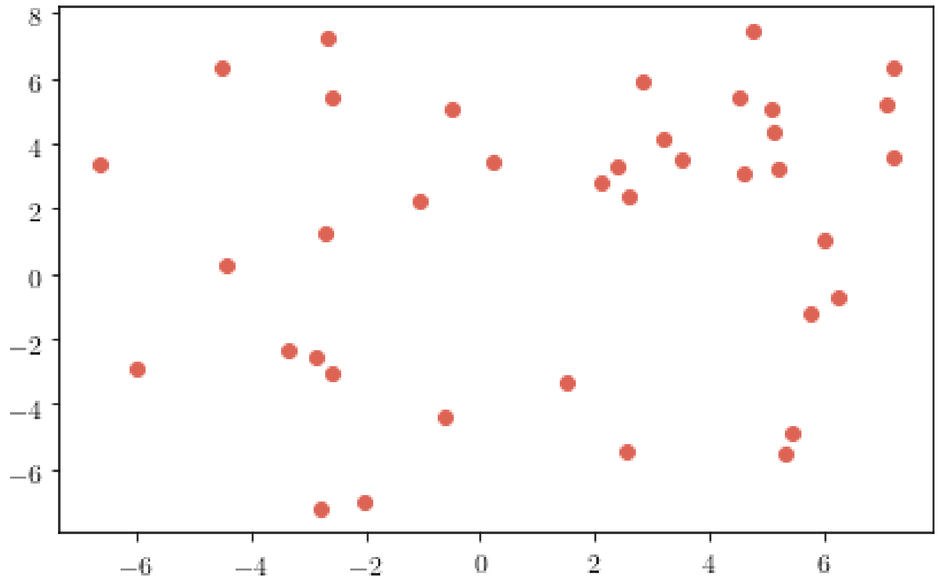
- Rate-of-file function:

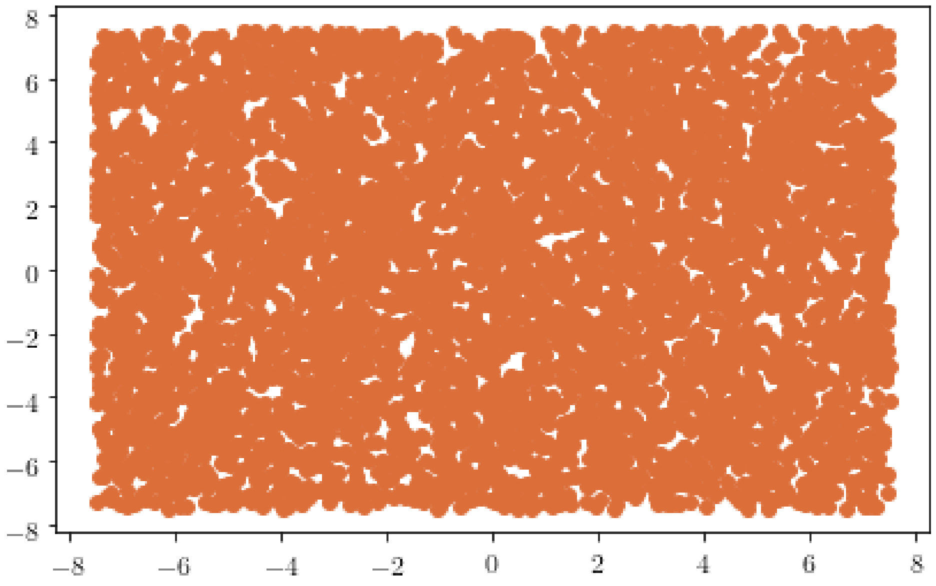
$$R(x, \Phi_t) = \frac{l(R)}{\mathcal{N}_0 + I(x, \Phi_t)}$$

- Sojourn time of point located at x_p:

$$d_p = \inf\left(t > t_p: \int_{t_p}^{u} R\left(x_p, \Phi_u\right) du > L_p \right)$$

Illustration





Two realizations: $\lambda = 0.05$ and $\lambda = 0.5$.
Left: after 200 000 iterations. Few users are present in the system and are gathered in small clusters
Right: after 10 000 iterations, users have filled up the whole region and transmission is nearly impossible

Markov properties

- Our point processes is a piece-wise constant counting measure and the dynamics are Markovian

$$(\Phi_t)_{t\geq 0} \text{ is a Markov jump process on } \mathcal{M}(\mathcal{D})$$

$$(\Phi_t(\mathcal{D}))_{t\geq 0} \text{ is a spatial birth-and-death process (SBD)}$$

- Transition probabilities:

$$\mathbb{P}(\text{birth at } x \text{ during } dt) = \lambda|\mathcal{D}|dt$$

$$\mathbb{P}(\text{death of } x \text{ during } dt) = \frac{1}{L}R(x, \Phi_t)dt$$

Goal : study the dynamics of the SBD

Mathematical framework

Main parameter for the dynamics: λ (arrival rate)

There exists an arrival rate λ_c such that:

- If $\lambda < \lambda_c$, the chain is stable (positive recurrent)
- If $\lambda > \lambda_c$, the system is unstable (transient)

Main result :

$$\lambda_c = \frac{\ell(R)}{L\|\ell\|}$$

Stochastic geometry tools

Palm calculus:

Φ_0 stationary point process (invariant by translation)

$\mathbb{E}^0_{\Phi_0}[\cdot]$: average conditioned by the events

ex : $\mathbb{E}^0_{\Phi_0}[I(0,\Phi_0)]$: interference experienced by the typical user

Powerful tool to reduce random sums:

$$\mathbb{E}\left[\sum_{x\in\Phi_0} f(x,\Phi_0)\right] = \mathbb{E}^0_{\Phi_0}[f(0,\Phi_0)]\mathbb{E}[\Phi_0(\mathcal{D})]$$

Stochastic geometry tools

Campbell's Theorem:

N PP of intensity measure Λ

$$\mathbb{E}\left[\sum_{x \in N} f(x)\right] = \int_{\mathbb{R}^d} f(x) \Lambda(dx)$$

Stochastic geometry tools

Miyazawa's Rate Conservation Law (RCL):

$\left(Y(t)\right)_{t\geq 0}$ real-valued stochastic process, right continuous with left limits, N PP of intensity λ and $\left(Y'(t)\right)_{t\geq 0}$ such that:

$$Y(1) = Y(0) + \int_0^1 Y'(s)ds + \int_0^1 \left(Y(s) - Y(s^-)\right)N(ds)$$

Then:

$$\mathbb{E}[Y'(0)] + \lambda \mathbb{E}_N^0[Y(0) - Y(0^-)] = 0$$

Proof of the main result

We place ourselves in the stationary framework and use the RCL 3 times:

- 1st use: to the number of transmitters

  We state that the number of points alive is the number of points alive minus the number of points that have died

$$\lambda|S| = \lambda_d$$

λ_d: intensity measure of the point process of death instants

Proof of the main result

- 2nd use: to the information present in the system

  Information present in the system is equal to the information that arrived minus the information processed

$$\lambda|S|L = \mathbb{E}\left[\sum_{x\in\Phi_0} R(x,\Phi_0)\right] = \mathbb{E}^0_{\Phi_0}[R(0,\Phi_0)]\mathbb{E}[\Phi_0(\mathcal{D})]$$

Proof of the main result

- 3rd use: to the interference experienced in the system

  Apply the RCL to the stochastic process $I_t = \sum_{x \in \Phi_t} I(x, \Phi_t)$

$$\lambda |S| \mathbb{E}^{\uparrow}[\mathcal{J}] = \lambda_d \mathbb{E}^{\downarrow}[\mathcal{L}]$$

$$\mathbb{E}^{\uparrow}[\mathcal{J}] = \mathbb{E}^{\downarrow}[\mathcal{L}]$$

$\mathcal{J} = \mathrm{I}_{0^+} - \mathrm{I}_0, \mathcal{L} = I_0 - I_{0^-}$ and the Palm expectations associated with arrival process and departure process

Proof of the main result

Lastly, compute $\mathbb{E}^{\uparrow}[\mathcal{J}]$ and $\mathbb{E}^{\downarrow}[\mathcal{L}]$:

- Increase in interference caused by an arrival in x: $+2I(x, \Phi_t)$

$$\mathbb{E}^{\uparrow}[\mathcal{J}] = 2\mathbb{E}[I(0, \Phi_0)] = 2\mathbb{E}\left[\sum_{x \in \Phi_0} \ell(\|x\|)\right] = \frac{2\mathbb{E}[\Phi_0(\mathcal{D})]\|\ell\|}{|S|}$$

- Decrease in interference caused by a departure in x: $-2I(x, \Phi_t)$

Caution: use the death process

$$\mathbb{E}^{\downarrow}[\mathcal{L}] = \frac{2\mathbb{E}^0_{\Phi_0}[R(0, \Phi_0)I(0, \Phi_0)]}{\lambda|S|L}\mathbb{E}[\Phi_0(\mathcal{D})]$$

Proof of the main result

Finally, we get:

$$\|\ell\| = \frac{\mathbb{E}^0_{\Phi_0}[R(0,\Phi_0)I(0,\Phi_0)]}{L\lambda}$$

Since $\mathbb{E}^0_{\Phi_0}[R(0,\Phi_0)I(0,\Phi_0)] \leq \ell(R)$, we get:

$$\lambda_c \leq \frac{\ell(R)}{L\|\ell\|}$$

Proof of the main result

To get the reciprocal:

- Use a discretization of the space in squares of area ε

- Build a new stochastic process that stochastically dominates (i.e. if it is stable, our dynamics are stable)

  Outline: if $I(x, \Phi') \geq I(x, \Phi)$ and Φ' is stable, Φ is stable

- Prove the stability of the dominating process using the fluid limit theory

Simulation setup

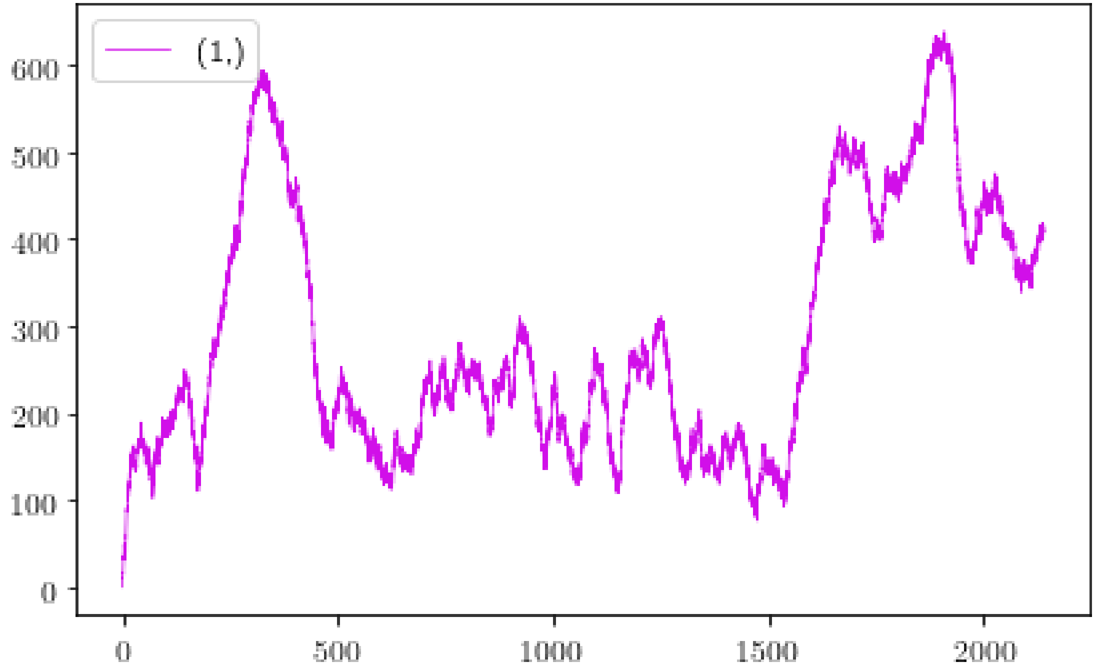
For a Markov jump process, interevent times are exponentially distributed

- Birth process: $t \sim \mathcal{E}\left(\frac{1}{\lambda|S|}\right)$
- Death process: for each $x \in \Phi_t$, $t_x \sim \mathcal{E}\left(\frac{L}{R(x,\Phi_t)}\right)$

Simulation parameters:

- $\mathcal{D} = [-7.5, 7.5]^2$
- $\ell(x) = \begin{cases} \frac{1}{1+x^4} & \text{if } x < 2 \\ 0 & \text{else} \end{cases}$
- $R = 1$
- $L = 1$

Proof of the main result



Realization for $\lambda = 0.1$ (close to the cutoff rate)

Evolution of $\Phi_t(\mathcal{D})$

-> We can observe excursions, which can go quite high

As λ gets closer to λ_c, the excursions will go higher

Extension of the problem: random channel fading

Transmissions can be impacted by channel fading (noise, etc.)

Add a fading coefficient $h_{xy} \sim \mathcal{E}(1)$ for each link (x, y)

$$R(x, \Phi_t) = \mathbb{E}_h \left[\frac{h_x l(R)}{\mathcal{N}_0 + \sum_{y \in \Phi_t \setminus \{x\}} h_{xy} \ell(\|x - y\|)} \right]$$

All results can be generalized to this case

Extensions of the problem : multivariate case

Use service differentiation:

- K orthogonal bands are available to transmit
- A user arriving in the system picks a number $1 \leq i \leq K$ of bands according to a given distribution $(p_i)_{1 \leq i \leq K}$
- He then picks uniformly at random the i bands used to transmit

Different dynamics:

- Users transmitting on disjoint sets of bands will ignore each other
- Users using a lot of bands will slow the traffic for other users… but will spend less time in the system

Extensions of the problem : multivariate case

For a user of type i:

$$I(x, \Phi_t) = \sum_{y \in \Phi_t \setminus \{x\}} \left| \mathcal{T}_x \cap \mathcal{T}_y \right| \ell(\|x - y\|)$$

$$R(x, \Phi_t) = \frac{i \ell(R)}{\mathcal{N}_0 + I(x, \Phi_t)}$$

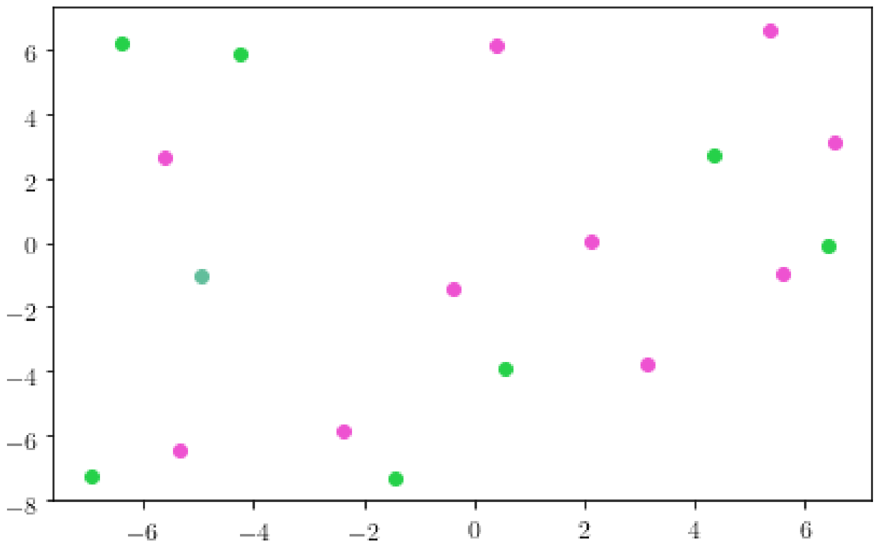
Extensions of the problem : multivariate case

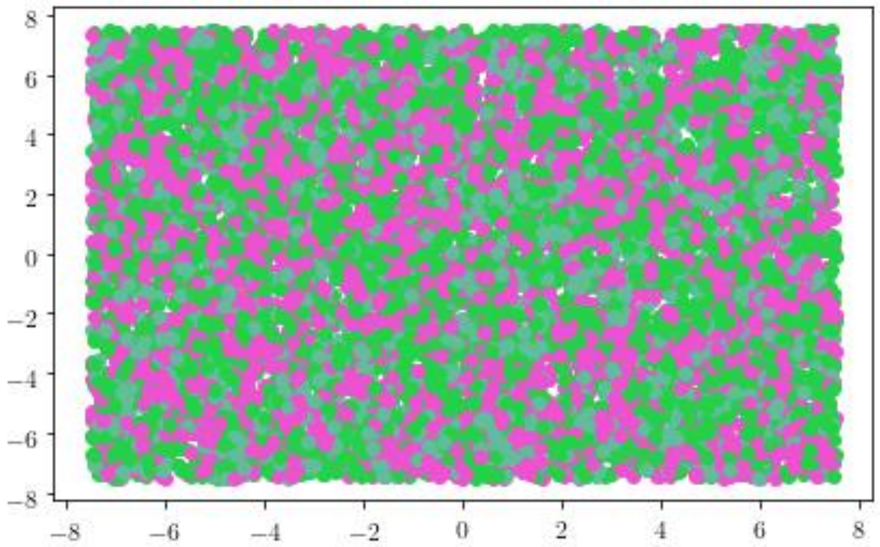
The equations are more complex but the tools used stay the same

Generalized cut-off arrival rate (conjecture):

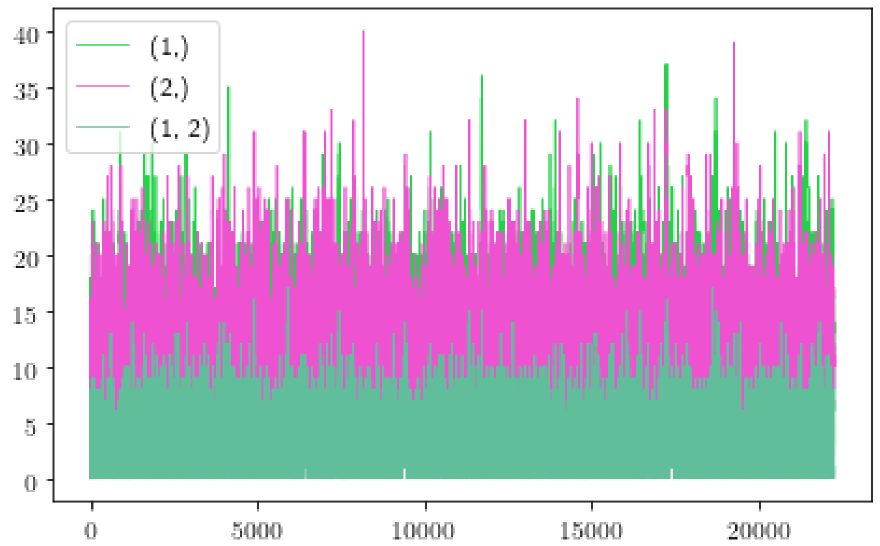
$$\lambda_c = \frac{\ell(R)}{L\|\ell\|} \frac{K}{p_1 + 2p_2 + \cdots + Kp_K}$$

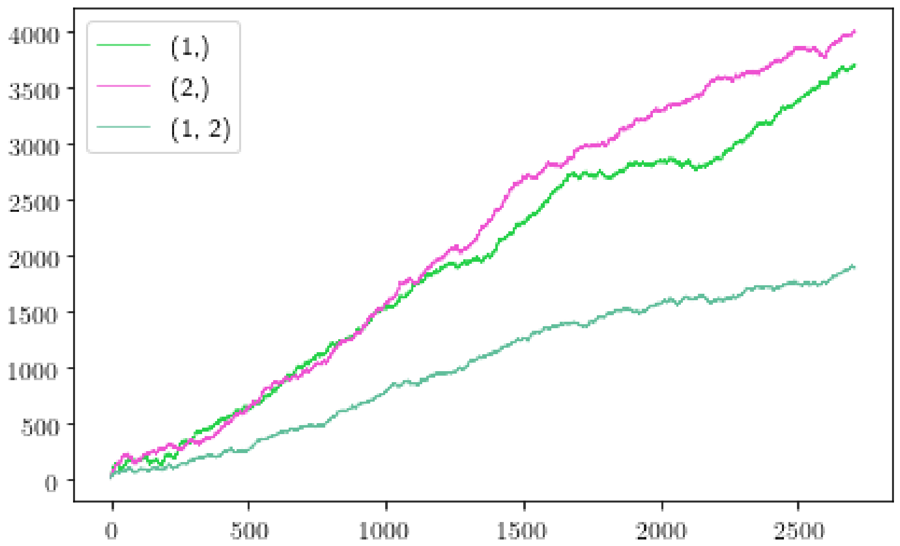
With $K = 2, p_1 = 0.8$ and $p_2 = 0.2, \lambda_c \approx 0.2001$





$\lambda = 0.05$ $\lambda = 0.22$



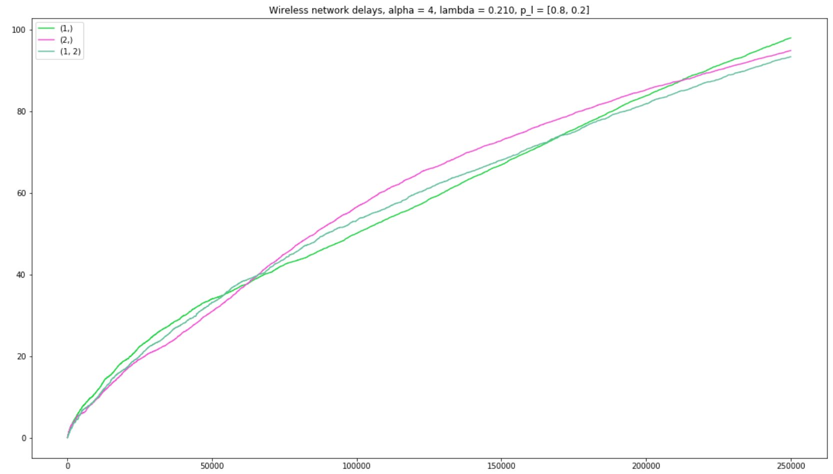


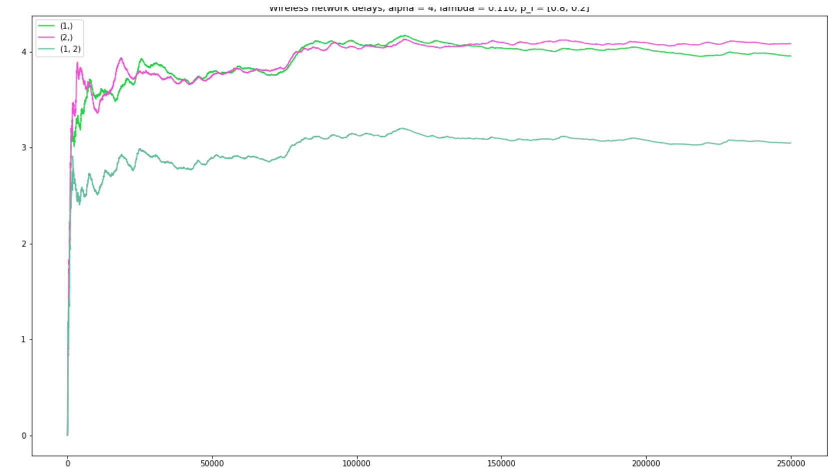
Extension of the problem : mulativariate case

Last interesting quantity to look at: mean sojourn time in the system.

Linked to the density of the point processes (Little's Law):

$$\tau = \frac{\mu_i}{\lambda p_i}$$





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Thank you for your attention