Large Poisson Games

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Based on Matías Núñez, “Approval Voting in Large Electorates”, in *Handbook on Approval Voting*, Springer-Verlag, 2010.

Reading Group
“Network Theory”

Lincs
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Position of the problem

Example: Voting

- Most of times, my action has no impact at all!
- If the population is fixed and other players act deterministically, then generally, any possible action is a best response for me (because it does not matter!).
- This is a difficulty to define equilibria.

If there is uncertainty on the population of players:

- There is always a small probability that a pivotal situation arises, where my action matters.
- I can choose my action based on these very unlikely events.

Poisson games is just a possible model to introduce population uncertainty (but it offers important practical advantages for mathematical tractability).

Overview

Principle:

1. Each player believes that the other players will use some strategy.
2. She compute the (unlikely) events where her action makes a difference.
3. She choose her strategy as a best response to this analysis.

Equilibria are defined as fixed points of this process.

References

- Roger Myerson (2000). Large Poisson games. *Journal of Economic Theory*, 94, 7–45.
  - Proves existence of equilibria
  - Magnitude Theorem
- Roger Myerson (2002). Comparison of scoring rules in Poisson voting games. *Journal of Economic Theory*, 103, 219–251.
  - Dual Magnitude Theorem
- Matías Núñez (2010). Condorcet consistency of approval voting: A counter example on large Poisson games. *Journal of Theoretical Politics*, 22(1), 64–84.
  - Magnitude Equivalence Theorem
- Matías Núñez (2010). Approval Voting in Large Electorates, in *Handbook on Approval Voting*, Springer-Verlag, 2010.
  - Reader's digest of all the above (+ alternative models)

Plan

Model

Direct probability calculation

Magnitude Theorem (Myerson, 2000)

Dual Magnitude Theorem (Myerson, 2002)

Magnitude Equivalence Theorem (Núñez 2010)

Conclusion

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Our running example: an election in Approval voting

Approval voting:

- Each voter votes for (= approves of) any number of candidates.
- The score of a candidate is the number of votes she receives.
- The candidate with highest score is declared the winner.
- In case of tie, the winner is chosen uniformly at random among the candidates with highest score.

Drawing the population of voters (= players)

- Expected number of voters: n.
- Actual number of voters: $N \sim \mathcal{P}(n)$ (Poisson distribution with mean n).

$$\mathbb{P}[N = k] = e^{-n} \frac{n^k}{k!}$$

Drawing the types of the voters

Each voter's type is independently drawn.

Type t	t_1	t_2	t_3
Preference ranking	α β γ	β α γ	γ α β
Type distribution $r(t)$	0.1	0.6	0.3
Actual number of voters $N(t)$	$\sim \mathcal{P}(0.1n)$	$\sim \mathcal{P}(0.6n)$	$\sim \mathcal{P}(0.3n)$

- Random variables $N(t)$ are independent.
- Types also have utilities (not written in the above table). For example, $u_{t_1}(\alpha) > u_{t_1}(\beta) > u_{t_1}(\gamma)$.

Choosing ballots ($=$ actions)

For example, consider this strategy function: $\begin{cases} \sigma(\alpha \mid t_1) = 1 \\ \sigma(\alpha\beta \mid t_2) = 1 \\ \sigma(\gamma \mid t_3) = 1 \end{cases}$.

- σ is generally not given. The issue will precisely be to find a σ that yields an equilibrium.

Then we have:

Ballot c	α	$\alpha\beta$	γ
Ballot distribution $\tau(c)$	0.1	0.6	0.3
Actual number of ballots $X(c)$	$\sim \mathcal{P}(0.1n)$	$\sim \mathcal{P}(0.6n)$	$\sim \mathcal{P}(0.3n)$

- Random variables $X(c)$ are independent.
- What happens if $\sigma(\alpha\beta \mid t_1) = 1$ instead?

Computing scores

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

Scores:

Candidate κ	α	β	γ
Score distribution $\rho(\kappa)$	0.7	0.6	0.3
Actual score $S(\kappa)$	$\sim \mathcal{P}(0.7n)$	$\sim \mathcal{P}(0.6n)$	$\sim \mathcal{P}(0.3n)$

- Are random variables $S(\kappa)$ independent?
- The winner is candidate α.

Advantages of the Poisson model

Common public information = *Environmental equivalence*

From the point of view of any voter, the number of other voters follows $\mathcal{P}(n)$, the number of other voters with type t follows $\mathcal{P}(nr(t))$, etc. Hence all voters live in the same environment, which is the same as seen by an external observer.

Independence of actions

The number $X(c)$ of voters who choose a given ballot is independent from the number of voters who choose another ballot.

Large Poisson games

We consider a sequence of Poisson games, parametrized by the expected number of players n.
Limit properties when $n \to \infty$?

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Magnitude Theorem (Myerson, 2000)

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Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

Direct probability calculation: Ex. 1

Event B_n: there is no ballot γ.

$$\mathbb{P}[X \in B_n] = \mathbb{P}[X(\gamma) = 0] = e^{-n\tau(\gamma)} \frac{(n\tau(\gamma))^0}{0!} = e^{-n\tau(\gamma)}$$

The **magnitude** of $B = (B_n)_{n \in \mathbb{N}}$ is defined as the coefficient in the exponent:

$$\mu(B) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}[X \in B_n] = -\tau(\gamma)$$

- A magnitude is always ≤ 0.
- When $n \to \infty$, it is **unlikely** that there is no ballot γ: $e^{-0.3n}$.
- But it is **infinitely less likely** that there is no ballot $\alpha\beta$: $e^{-0.6n}$.
- Similarly, it is **infinitely more likely** that there is no ballot α: $e^{-0.1n}$.

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

Direct probability calculation: Ex. 2

Event B_n: there are just as many ballots $\alpha\beta$ as γ.

$$\begin{aligned}
\mathbb{P}[X \in B_n] &= \sum_{k=0}^{\infty} \mathbb{P}[X(\alpha\beta) = k \text{ and } X(\gamma) = k] \\
&= \sum_{k=0}^{\infty} \mathbb{P}[X(\alpha\beta) = k] \cdot \mathbb{P}[X(\gamma) = k] \\
&= e^{-n(\tau(\alpha\beta)+\tau(\gamma))} \sum_{k=0}^{\infty} \frac{\left(n^2\tau(\alpha\beta)\tau(\gamma)\right)^k}{(k!)^2} \\
&= e^{-n(\tau(\alpha\beta)+\tau(\gamma))} I_0\left(2n\sqrt{\tau(\alpha\beta)\tau(\gamma)}\right) \\
&= \exp\left(-n(\tau(\alpha\beta)+\tau(\gamma)) + 2n\sqrt{\tau(\alpha\beta)\tau(\gamma)} + o(n)\right)
\end{aligned}$$

$$\mu(B) = -\tau(\alpha\beta) - \tau(\gamma) + 2\sqrt{\tau(\alpha\beta)\tau(\gamma)} = -\left(\sqrt{\tau(\alpha\beta)} - \sqrt{\tau(\gamma)}\right)^2$$

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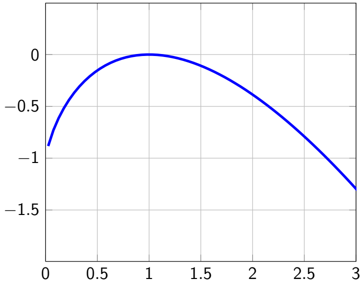
Magnitude of a sequence of points

b_n sequence of *points*. For each kind of ballot c, it specifies how many ballots $b_n(c)$.

$$
\begin{aligned}
\mu(b) &= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[X = b_n] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \prod_{c\in\mathcal{C}} \mathbb{P}[X(c) = b_n(c)] \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{c\in\mathcal{C}} \log \left(e^{-n\tau_n(c)} \frac{\left(n\tau_n(c)\right)^{b_n(c)}}{b_n(c)!} \right) \\
&= \lim_{n\to\infty} \sum_{c\in\mathcal{C}} \tau_n(c) \left(\frac{b_n(c)}{n\tau_n(c)} \left(1 - \log \frac{b_n(c)}{n\tau_n(c)} \right) - 1 \right)
\end{aligned}
$$

Offset



Hence we have:

$$\mu(b) = \lim_{n\to\infty} \sum_{c\in\mathcal{C}} \tau_n(c)\psi\left(\frac{b_n(c)}{n\tau_n(c)}\right),$$

where $\psi(x) = x(1 - \log x) - 1$ and $\psi(0) = -1$.

We define the **offset** of ballot c in this sequence of points b_n as:

$$\phi_c = \lim_{n\to\infty} \frac{b_n(c)}{n\tau_n(c)}.$$

It is the limit ratio between the number of actual ballots c in this particular sequence of points and what would be expected in general.

Magnitude Theorem

Let (B_n) be a sequence of outcomes whose magnitude is defined. Then:

$$\begin{aligned}\mu(B) &= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[X \in B_n] \\ &= \lim_{n\to\infty} \max_{b_n \in B_n} \frac{1}{n} \log \mathbb{P}[X = b_n]\end{aligned}$$

I.e. if b_n is defined as the most probable point in B_n, then $\mu(B) = \mu(b)$. In practice:

$$\mu(B) = \lim_{n\to\infty} \max_{b_n \in B_n} \sum_{c \in \mathcal{C}} \tau_n(c) \psi\left(\frac{b_n(c)}{n\tau_n(c)}\right)$$

N.B.: We define the offset of ballot c in B as its offset in b. [In fact, this offset is the same in any sequence of points extracted from (B_n) that has the same magnitude.]

Magnitude Theorem: Ex. 1

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

B_n: there is no ballot γ.
Any point $b_n \in B_n$ is of the form: $X(\alpha) = k, X(\alpha\beta) = k', X(\gamma) = 0$.

$$\mu(B) = \lim_{n\to\infty} \max_{b_n \in B_n} \sum_{c\in\mathcal{C}} \tau_n(c)\psi\left(\frac{b_n(c)}{n\tau_n(c)}\right)$$

$$= \lim_{n\to\infty} \max_{k,k'\in\mathbb{N}} \tau(\alpha)\underbrace{\psi\left(\frac{k}{n\tau(\alpha)}\right)}_{\text{lim max}=0} + \tau(\alpha\beta)\underbrace{\psi\left(\frac{k'}{n\tau(\alpha\beta)}\right)}_{\text{lim max}=0} + \tau(\gamma)\underbrace{\psi\left(\frac{0}{n\tau(\gamma)}\right)}_{=-1}$$

$$= -\tau(\gamma)$$

Moreover: $\phi_\alpha = \phi_{\alpha\beta} = 1$ and $\phi_\gamma = 0$.

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

Magnitude Theorem: Ex. 2

B_n: there are just as many ballots $\alpha\beta$ as γ.

$$\begin{aligned}
\mu(B) &= \lim_{n\to\infty} \max_{b_n \in B_n} \sum_{c\in\mathcal{C}} \tau_n(c)\psi\left(\frac{b_n(c)}{n\tau_n(c)}\right) \\
&= \lim_{n\to\infty} \max_{k,k'\in\mathbb{N}} \underbrace{\tau(\alpha)\,\psi\left(\frac{k}{\tau(\alpha)n}\right)}_{\lim\max=0} + \tau(\alpha\beta)\psi\left(\frac{k'}{\tau(\alpha\beta)n}\right) + \tau(\gamma)\psi\left(\frac{k'}{\tau(\gamma)n}\right) \\
&= \lim_{n\to\infty} \max_{k'\in\mathbb{N}} \tau(\alpha\beta)\psi\left(\frac{k'}{\tau(\alpha\beta)n}\right) + \tau(\gamma)\psi\left(\frac{k'}{\tau(\gamma)n}\right) \\
&= \max_{x\geq 0} \tau(\alpha\beta)\psi\left(\frac{x}{\tau(\alpha\beta)}\right) + \tau(\gamma)\psi\left(\frac{x}{\tau(\gamma)}\right) \\
&= -\left(\sqrt{\tau(\alpha\beta)} - \sqrt{\tau(\gamma)}\right)^2
\end{aligned}$$

Magnitude Theorem: Ex. 2 (continued)

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

And the offsets:

$$\begin{cases} \phi_\alpha = 1 & \Rightarrow \quad b_n(\alpha) \sim 0.1n \\ \phi_{\alpha\beta} = \sqrt{\tau(\gamma)/\tau(\alpha\beta)} & \Rightarrow \quad b_n(\alpha\beta) \sim n\sqrt{\tau(\alpha\beta)\tau(\gamma)} \simeq 0.42n \\ \phi_\gamma = \sqrt{\tau(\alpha\beta)/\tau(\gamma)} & \Rightarrow \quad b_n(\gamma) \sim n\sqrt{\tau(\alpha\beta)\tau(\gamma)} \simeq 0.42n \end{cases}$$

Remark: the total number of voters in b_n is not n.

Myerson 2000: bonus tracks (explained with the hands)

Offset theorem

Assume you have studied $(B_n)_{n\in\mathbb{N}}$, e.g., $B_n : X(\alpha\beta) = X(\gamma)$.
Consider a "finite translation", e.g. $B'_n : X(\alpha\beta) = X(\gamma) + 1$.
The offset theorem gives a easy way to compute:

$$\lim_{n\to\infty} \frac{\mathbb{P}(B'_n)}{\mathbb{P}(B_n)}.$$

[In particular, B and B' have the same magnitude.]

Hyperplane theorem

Quite technical, but the main consequence is:
For events **whose probability does not tend to 0**, you can approximate by a normal distribution: $X(c) \sim \mathcal{N}(n\tau(c), \sqrt{n\tau(c)})$.

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DMT

Let B be an outcome defined by a finite set of linear inequalities:

$$\begin{cases} a_1(c)X(c) + a_1(c')X(c') + \ldots \geq 0 \\ \vdots \\ a_K(c)X(c) + a_K(c')X(c') + \ldots \geq 0 \end{cases}$$

Suppose that $\lambda_1, \ldots, \lambda_K \geq 0$ is an argmin of:

$$F(\lambda) = \sum_{c \in \mathcal{C}} \tau(c) \left[\exp\left(\sum_{k \leq K} \lambda_k a_k(c) \right) - 1 \right]$$

Then $\mu(B) = F(\lambda)$. Moreover, for any ballot c:

$$\phi_c = \exp\left(\sum_{k \leq K} \lambda_k a_k(c) \right)$$

DMT: Ex. 1

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

B_n: there is no ballot γ.
Only one constraint:

$$0X(\alpha) + 0X(\alpha\beta) - 1X(\gamma) \geq 0.$$

$$F(\lambda) = \tau(\alpha)\underbrace{\left[e^{0\lambda_1} - 1\right]}_{=0} + \tau(\alpha\beta)\underbrace{\left[e^{0\lambda_1} - 1\right]}_{=0} + \tau(\gamma)\underbrace{\left[e^{-1\lambda_1} - 1\right]}_{\to -1 \text{ if } \lambda_1 \to \infty}$$

$$\mu(B) = -\tau(\gamma)$$

$$\begin{cases} \phi_\alpha = e^{0\lambda_1} = 1 \\ \phi_{\alpha\beta} = e^{0\lambda_1} = 1 \\ \phi_\gamma = e^{-1\lambda_1} = 0 \end{cases}$$

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

DMT: Ex. 2

B_n: there are just as many ballots $\alpha\beta$ as γ.

Constraints:

$$\begin{cases} 0X(\alpha) + 1X(\alpha\beta) - 1X(\gamma) \geq 0 \\ 0X(\alpha) - 1X(\alpha\beta) + 1X(\gamma) \geq 0 \end{cases}$$

$$F(\lambda) = \tau(\alpha)\underbrace{\left[e^{0\lambda_1+0\lambda_2} - 1\right]}_{=0} + \tau(\alpha\beta)\left[e^{1\lambda_1-1\lambda_2} - 1\right] + \tau(\gamma)\left[e^{-1\lambda_1+1\lambda_2} - 1\right]$$

For $e^{\lambda_1-\lambda_2} = \sqrt{\tau(\gamma)/\tau(\alpha\beta)}$, we obtain:

$$\mu(B) = 2\sqrt{\tau(\alpha\beta)\tau(\gamma)} - \tau(\alpha\beta) - \tau(\gamma) = -\left(\sqrt{\tau(\alpha\beta)} - \sqrt{\tau(\gamma)}\right)^2$$

And we could deduce the offsets immediately.

DMT: About the proof

- In Myerson 2002, the proof is very short.
- But it is a bit obscure where the function $F(\lambda)$ comes from.
- I will try to provide a clue about this!

Kuhn-Tucker conditions: one constraint

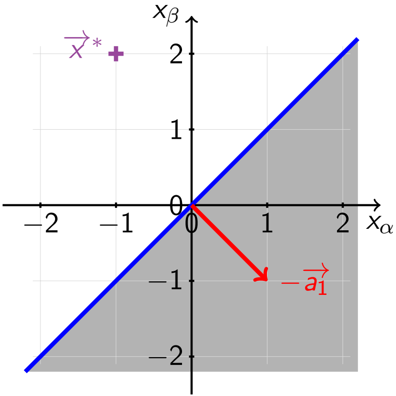
$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & a_1(\alpha)x_\alpha + a_1(\beta)x_\beta \geq 0 \end{array}$$

Example:

$-x_\alpha + x_\beta \geq 0$

If $\overrightarrow{x}^*$ is in the interior of the cone:

$$\overrightarrow{\nabla} f(\overrightarrow{x}^*) = \overrightarrow{0}$$



Kuhn-Tucker conditions: one constraint

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & a_1(\alpha)x_\alpha + a_1(\beta)x_\beta \geq 0 \end{array}$$

Example:

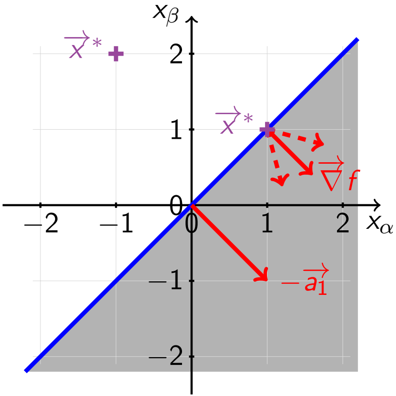
$-x_\alpha + x_\beta \geq 0$

If $\overrightarrow{x}^*$ is in the interior of the cone:

$$\overrightarrow{\nabla} f(\overrightarrow{x}^*) = \overrightarrow{0}$$

If $\overrightarrow{x}^*$ is on the frontier of the cone:

$$\overrightarrow{\nabla} f(\overrightarrow{x}^*) = -\lambda_1 \overrightarrow{a_1} \quad (\text{with } \lambda_1 \geq 0)$$



Kuhn-Tucker conditions: one constraint

max $\quad f(x)$
s.t. $\quad a_1(\alpha)x_\alpha + a_1(\beta)x_\beta \geq 0$

Example:
$-x_\alpha + x_\beta \geq 0$

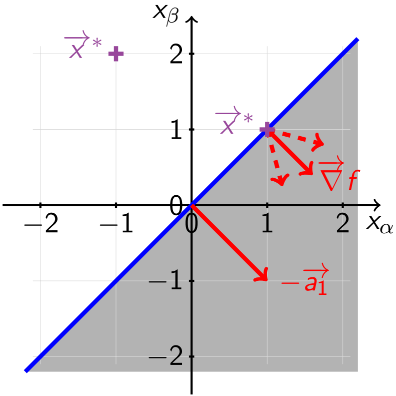
If $\overrightarrow{x}^*$ is in the interior of the cone:

$$\overrightarrow{\nabla} f(\overrightarrow{x}^*) = \overrightarrow{0}$$

If $\overrightarrow{x}^*$ is on the frontier of the cone:

$$\overrightarrow{\nabla} f(\overrightarrow{x}^*) = -\lambda_1 \overrightarrow{a_1} \quad \text{(with } \lambda_1 \geq 0)$$

Anyway, the second condition is met.
Moreover, if $\lambda_1 > 0$, then $\overrightarrow{a_1} \cdot \overrightarrow{x} = 0$.



Kuhn-Tucker conditions: several constraints

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & a_1(\alpha)x_\alpha + a_1(\beta)x_\beta \geq 0 \\ & a_2(\alpha)x_\alpha + a_2(\beta)x_\beta \geq 0 \end{array}$$

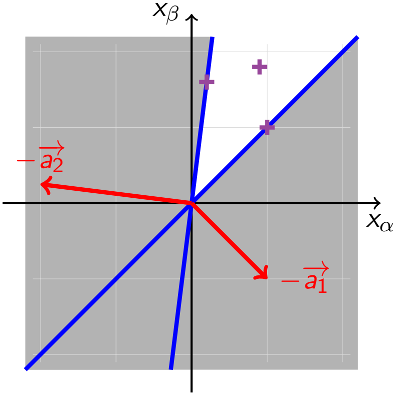
$\overrightarrow{x}^*$ interior: $\overrightarrow{\nabla} f(\overrightarrow{x}^*) = \overrightarrow{0}$.

$\overrightarrow{x}^*$ on first frontier: $\overrightarrow{\nabla} f(\overrightarrow{x}^*) = -\lambda_1 \overrightarrow{a_1}$.

$\overrightarrow{x}^*$ on second frontier: $\overrightarrow{\nabla} f(\overrightarrow{x}^*) = -\lambda_2 \overrightarrow{a_2}$.

Example:

$$\begin{array}{rl} -x_\alpha + \quad\quad x_\beta & \geq 0 \\ 2x_\alpha - 0.25x_\beta & \geq 0 \end{array}$$



Kuhn-Tucker conditions: several constraints

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & a_1(\alpha)x_\alpha + a_1(\beta)x_\beta \geq 0 \\ & a_2(\alpha)x_\alpha + a_2(\beta)x_\beta \geq 0 \end{array}$$

Example:

$$\begin{array}{r} -x_\alpha + \quad x_\beta \geq 0 \\ 2x_\alpha - 0.25x_\beta \geq 0 \end{array}$$

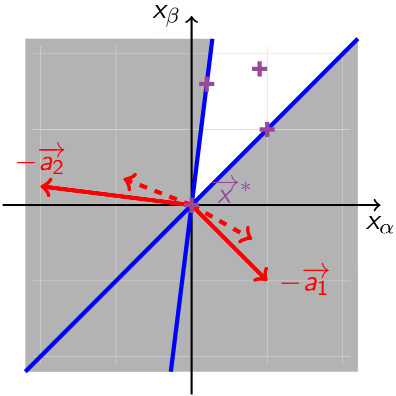
$\vec{x}^*$ interior: $\vec{\nabla} f(\vec{x}^*) = \vec{0}$.

$\vec{x}^*$ on first frontier: $\vec{\nabla} f(\vec{x}^*) = -\lambda_1 \vec{a_1}$.

$\vec{x}^*$ on second frontier: $\vec{\nabla} f(\vec{x}^*) = -\lambda_2 \vec{a_2}$.

If $\vec{x}^*$ is on the intersection of frontiers:

$$\vec{\nabla} f(\vec{x}^*) = -\lambda_1 \vec{a_1} - \lambda_2 \vec{a_2} \quad \text{(with } \lambda_k \geq 0\text{)}.$$



In all cases, the above condition is met.

Moreover, if $\lambda_k > 0$, then $\vec{a_k} \cdot \vec{x} = 0$.

Applying Kuhn-Tucker conditions in our case

$$\begin{array}{ll} \max & f(\phi) = \tau(\alpha)\psi(\phi_\alpha) + \tau(\beta)\psi(\phi_\beta) \\ \text{s.t.} & a_1(\alpha)\tau(\alpha)\,\phi_\alpha + a_1(\beta)\tau(\beta)\,\phi_\beta \geq 0 \\ & a_2(\alpha)\tau(\alpha)\,\phi_\alpha + a_2(\beta)\tau(\beta)\,\phi_\beta \geq 0 \end{array}$$

Condition on $\vec{\nabla} f$:

$$\begin{pmatrix} -\tau(\alpha)\log\phi_\alpha \\ -\tau(\beta)\log\phi_\beta \end{pmatrix} = -\lambda_1 \begin{pmatrix} a_1(\alpha)\tau(\alpha) \\ a_1(\beta)\tau(\beta) \end{pmatrix} - \lambda_2 \begin{pmatrix} a_2(\alpha)\tau(\alpha) \\ a_2(\beta)\tau(\beta) \end{pmatrix}$$

$$\phi_\alpha = e^{\lambda_1 a_1(\alpha) + \lambda_2 a_2(\alpha)} \quad (\text{resp. } \beta)$$

When $\lambda_k > 0$, then the point is on the corresponding frontier. For example:

$$a_1(\alpha)\tau(\alpha)e^{\lambda_1 a_1(\alpha)+\lambda_2 a_2(\alpha)} + a_1(\beta)\tau(\beta)e^{\lambda_1 a_1(\beta)+\lambda_2 a_2(\beta)} = 0$$

$$\frac{\partial}{\partial \lambda_1}\left[\tau(\alpha)e^{\lambda_1 a_1(\alpha)+\lambda_2 a_2(\alpha)} + \tau(\beta)e^{\lambda_1 a_1(\beta)+\lambda_2 a_2(\beta)}\right] = 0$$

Applying Kuhn-Tucker conditions in our case

$$\begin{array}{ll} \max & f(\phi) = \tau(\alpha)\psi(\phi_\alpha) + \tau(\beta)\psi(\phi_\beta) \\ \text{s.t.} & a_1(\alpha)\tau(\alpha)\,\phi_\alpha + a_1(\beta)\tau(\beta)\,\phi_\beta \geq 0 \\ & a_2(\alpha)\tau(\alpha)\,\phi_\alpha + a_2(\beta)\tau(\beta)\,\phi_\beta \geq 0 \end{array}$$

Condition on $\vec{\nabla} f$:

$$\begin{pmatrix} -\tau(\alpha)\log\phi_\alpha \\ -\tau(\beta)\log\phi_\beta \end{pmatrix} = -\lambda_1 \begin{pmatrix} a_1(\alpha)\tau(\alpha) \\ a_1(\beta)\tau(\beta) \end{pmatrix} - \lambda_2 \begin{pmatrix} a_2(\alpha)\tau(\alpha) \\ a_2(\beta)\tau(\beta) \end{pmatrix}$$

$$\phi_\alpha = e^{\lambda_1 a_1(\alpha) + \lambda_2 a_2(\alpha)} \quad (\text{resp. } \beta)$$

When $\lambda_k > 0$, then the point is on the corresponding frontier. For example:

$$a_1(\alpha)\tau(\alpha)e^{\lambda_1 a_1(\alpha) + \lambda_2 a_2(\alpha)} + a_1(\beta)\tau(\beta)e^{\lambda_1 a_1(\beta) + \lambda_2 a_2(\beta)} = 0$$

$$\frac{\partial}{\partial \lambda_1} F(\lambda) = 0$$

Terminological remarks

- Dual cone: cone generated by the $\overrightarrow{a}_k$.
- λ_k: Lagrange multiplier associated to constraint k.
- $F(\lambda)$ is the Lagrange function (or Lagrangian) of the original optimization problem. It plays the role of objective function in the dual problem.

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Conclusion

MET

Rigorous definition of a pivot situation for subset Y of candidates:

$$\begin{cases} \forall \kappa \in Y, S(\kappa) \geq \max S - 1 \\ \forall \kappa \notin Y, S(\kappa) \leq \max S - 2 \end{cases}$$

These constraints are not linear!

We have $\mu[\text{pivot}(Y)] = \mu(B)$, where outcome B is defined by:

$$\begin{cases} \forall \kappa, \kappa' \in Y : & S(\kappa) = S(\kappa') \\ \forall \kappa \in Y, \kappa' \notin Y : & S(\kappa) \geq S(\kappa') \end{cases}$$

These constraints **are** linear! Hence we can apply DMT.

MET example: pivot α vs β

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

MET:

$$\begin{cases} S(\alpha) \geq S(\beta): & 1X(\alpha) \geq 0 \\ S(\beta) \geq S(\alpha): & -1X(\alpha) \geq 0 \\ S(\alpha) \geq S(\gamma): & 1X(\alpha) + 1X(\alpha\beta) - 1X(\gamma) \geq 0 \end{cases}$$

DMT:

$$F(\lambda) = \tau(\alpha)\left[e^{1\lambda_1 - 1\lambda_2 + 1\lambda_3} - 1\right] + \tau(\alpha\beta)\left[e^{1\lambda_3} - 1\right] + \tau(\gamma)\left[e^{-1\lambda_3} - 1\right]$$

Minimization: $\lambda_1 - \lambda_2 \to -\infty$ and $\lambda_3 = 0$ (we cannot do better because $\lambda_3 \geq 0$).
We obtain:

$$\mu[\text{pivot}(\alpha, \beta)] = -\tau(\alpha).$$

Ballot c	α	$\alpha\beta$	γ
$\tau(c)$	0.1	0.6	0.3

End of the example

$$\begin{cases} \mu[\text{pivot}(\alpha,\beta)] = -\tau(\alpha) = -0.1 \\ \mu[\text{pivot}(\alpha,\gamma)] = -\left(\sqrt{\tau(\alpha)+\tau(\alpha\beta)} - \sqrt{\tau(\gamma)}\right)^2 \simeq -0.08 \\ \mu[\text{pivot}(\beta,\gamma)] = -\tau(\alpha) - \left(\sqrt{\tau(\alpha\beta)} - \sqrt{\tau(\gamma)}\right)^2 \simeq -0.15 \end{cases}$$

$$\Rightarrow \mu[\text{pivot}(\alpha,\gamma)] > \mu[\text{pivot}(\alpha,\beta)] > \mu[\text{pivot}(\beta,\gamma)]$$

Type t	t_1	t_2	t_3
Preference ranking	α	β	γ
	β	α	α
	γ	γ	β
Best response	α	$\alpha\beta$	γ

$\Rightarrow$ It is an equilibrium!

Plan

Model

Direct probability calculation

Magnitude Theorem (Myerson, 2000)

Dual Magnitude Theorem (Myerson, 2002)

Magnitude Equivalence Theorem (Núñez 2010)

Conclusion

Large Poisson Games

- Models with variable population generate uncertainty, even if players have deterministic strategies.
- Players may base their strategy on very unlikely events.
- Poisson games $\Rightarrow$ environmental equivalence $\Rightarrow$ easier to handle!
- Large Poisson games $\Rightarrow$ reason in terms of magnitudes (instead of probabilities).
- Magnitude Theorem, DMT and MET gives practical tools to compute magnitudes.

Thank you!